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# Transonic shocks and free boundary problems for the full Euler equations in infinite nozzles

Gui-Qiang Chen [a,*], Jun Chen [b], Mikhail Feldman [b]

[a] *Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA*

[b] *Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706-1388, USA*

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**Abstract**

We establish the existence, stability, and asymptotic behavior of transonic flows with a transonic shock for the steady, full Euler equations in two-dimensional infinite nozzles of slowly varying cross-sections. Given a smooth incoming flow that is close to a uniform supersonic state at the entrance, we prove that there exists a transonic flow whose infinite downstream smooth subsonic region is separated by a smooth transonic shock from the upstream supersonic flow. The solution is unique within the class of transonic solutions that are close to the background solution. This problem is approached by a free boundary problem in which the transonic shock is formulated as a free boundary. An iteration scheme for the free boundary is developed and its fixed point is shown to exist, which is a solution of the free boundary problem, by combining some delicate estimates for a second-order nonlinear elliptic equation on a Lipschitz domain.


**Résumé**

Nous établissons l'existence, la stabilité, et le comportement asymptotique des écoulements transoniques avec un choc transonique pour les équations d'Euler complètes et indépendantes du temps dans des tuyères bidimensionelles, infinies et ayant des coupes transversales qui varient lentement. Etant donné un écoulement entrant régulier qui est proche d'un état supersonique uniforme à l'entrée, nous démontrons que—dans la direction de l'écoulement—il existe un écoulement transonique, dont la région (infinie) subsonique devant le choc est séparée de la région supersonique derrière le choc, par un choc transonique à travers une courbe régulière. La solution est unique dans la classe des solutions transoniques qui sont proches de la solution de base. Ce problème est abordé par un problème à frontière libre, dans lequel le choc est formulé comme frontière libre. Nous développons un schèma d'itération pour la frontière libre et, en combinant quelques estimations délicates pour une équation elliptique non linéaire de deuxième ordre sur un domaine lipschitzien, nous démontrons qu'il existe un point fixe solution du problème à frontière libre.


*Keywords:* Transonic shocks; Full Euler equations; Polytropic gas; Free boundary problems

* Corresponding author. Tel.: +1 847 491 5553; fax: +1 847 491 8906.
*E-mail addresses:* gqchen@math.northwestern.edu (G.-Q. Chen), jchen@math.wisc.edu (J. Chen), feldman@math.wisc.edu (M. Feldman).

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## 1. Introduction

We establish the existence, stability, and asymptotic behavior of transonic flows with a transonic shock for the full Euler equations in two-dimensional infinite nozzles of slowly varying cross-sections. The transonic flow is governed by the following two-dimensional steady, full Euler equations:

$$\begin{cases} \nabla \cdot (\rho \mathbf{u}) = 0, \\ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \\ \nabla \cdot (\rho \mathbf{u}(E + p/\rho)) = 0, \end{cases} \tag{1.1}$$

where $\nabla$ is the gradient in $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{u} = (u_1, u_2)$ is the velocity, $\rho$ the density, $p$ the pressure, and

$$E = \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{(\gamma - 1)\rho}$$

the energy with adiabatic exponent $\gamma > 1$. The sonic speed of the flow is:

$$c = \sqrt{\gamma p/\rho}.$$

The flow is subsonic if $|\mathbf{u}| < c$ and supersonic if $|\mathbf{u}| > c$. For a transonic flow, both cases occur in the flow where shocks are generically developed (cf. [23]).

Given an incoming smooth flow that is close to a uniform supersonic state (i.e. smooth Cauchy data) at the entrance (say $x_1 = -1$) and the subsonic condition in the smooth downstream region, we are interested in whether there exists a transonic flow whose infinite downstream subsonic region is separated by a smooth transonic shock from the upstream supersonic flow. This nozzle problem can be approached by a free boundary problem in which the transonic shock is formulated as a free boundary.

Such transonic problems have been studied under several different physical situations in the recent years. In Chen and Feldman [5–8], three nonlinear approaches have been developed to establish the existence and stability of transonic shocks for the multidimensional steady potential flow equation and applied to handling transonic flow problems in infinite channels and nozzles. Recently, Chen [11] also considered this problem in a bounded channel for the two-dimensional steady Euler flows with a certain symmetry and obeying the Bernoulli law with a uniform Bernoulli constant (also see [12]). The existence and uniqueness of transonic flows with a transonic shock were established in Chen, Chen and Song [4] for a *bounded* two-dimensional nozzle of slowly varying cross-sections with flat entrance section. Also see [13,26] for related bounded nozzle problems. There are related results for further simplified models: the unsteady transonic small disturbance equation in Canic, Keyfitz and Lieberman [1] and Canic, Keyfitz and Kim [2], the pressure-gradient system in Zheng [27,28], and the nonlinear wave system in Canic, Keyfitz and Kim in [3] (see the further references cited therein). Also see [9,10,24,25] for a program to deal with transonic and sonic-subsonic flows through (vanishing viscosity or relaxation) approximate or exact solutions via the method of compensated compactness, and see [18] where a smooth transition from subsonic to supersonic flow was studied.

In this paper, we systematically study the infinite transonic nozzle problem in the context of the full Euler equations. We prove that, for this nozzle problem, there exists a transonic flow whose infinite downstream smooth subsonic region is separated by a smooth transonic shock from the upstream supersonic flow. To achieve this, we first employ the coordinate transformation of Euler–Lagrange type so that the original streamlines in Eulerian coordinates become straight lines and the infinite nozzle in Eulerian coordinates becomes an infinite channel in the new Lagrangian coordinates. Then we use one of the new equations to identify a potential function $\phi$ in Lagrangian coordinates. By capturing the conservation properties of the Euler system, we derive a single second-order nonlinear elliptic equation for the potential function $\phi$ in the subsonic region so that the full Euler equations are reduced to this single second-order equation. The advantage of this approach is that, given the shock location, all the physical variables $(\mathbf{u}, p, \rho)$ can be expressed as functions of the gradient of $\phi$, and the asymptotic behavior $\phi_\infty$ of the potential $\phi$ at the infinite exit can be uniquely determined.

To solve the free boundary problem, we have to determine both the free boundary and the subsonic phase defined in the downstream region with the free boundary as a part of its boundary. We approach this problem by developing an iteration scheme via updating the location of the shock front and designing a corresponding iteration map. In order to define the map for the given shock location, we first linearize the second-order elliptic equation for the identified potential function based on the limit function $\phi_\infty$ of the potential $\phi$, solve the linearized problem in the fixed region, and then make delicate estimates of the solutions, especially the corner singularity near the intersection between the fixed shock and the nozzle boundaries. Finally, these estimates allow us to prove that the map is a contraction map so that the fixed point of the map is the real shock front and the corresponding subsonic solution in the downstream region is the real subsonic phase for the free boundary problem. Since the transformation between the Eulerian and Lagrangian coordinates is invertible, we obtain the existence and uniqueness of solutions of the infinite nozzle problem in Eulerian coordinates by transforming back the solutions in Lagrangian coordinates. The asymptotic behavior of solutions at the infinite exit is also clarified. The stability of transonic shocks and corresponding transonic flows is also established by employing the coordinate transformation of Euler–Lagrange type and careful, detailed estimates of the solutions.

Another advantage in our analysis here is in the context of the real full Euler equations so that the solutions do not necessarily obey Bernoulli's law with a *uniform* Bernoulli constant, i.e., the Bernoulli constant is allowed to change for different fluid trajectories (compare with the setup in [11–13]). Since we work on the infinite channel in the new Lagrangian coordinates by iterating the location of shock front and making estimates of the corresponding solution in the downstream region, we do not require additional symmetry of the solutions (unlike in [6,11]), or the flat condition of the entrance nozzle part (unlike in [4]). We remark that, by the closeness assumption of the solution $U$ to the uniform flow in the subsonic region, we obtain the asymptotic behavior of $U$ as $x_1 \to \infty$. The asymptotic state $U_\infty = (\mathbf{u}_\infty, p_\infty, \rho_\infty)$ is uniquely determined by the state $U_-$ of the incoming flow at the entrance $x_1 = -1$. In particular, the vertical component of the asymptotic velocity equals to zero, $u_{2\infty} = 0$, and the pressure $p_\infty$ is a constant determined by the incoming flow $U_-$. In general, $u_{1\infty}$ and $\rho_\infty$ are not constants, which are actually functions of $x_2$; and the pressure condition at the exit of the nozzle is ill-posed (cf. [6–8,14]).

The organization of this paper is as follows. In Section 2, we formulate the transonic nozzle problem into a free boundary problem in the two-dimensional infinite nozzle and state the main theorems. In Section 3, we introduce a coordinate transformation of Euler–Lagrange type and reformulate the free boundary problem in the new coordinates; then we identify a potential function $\phi$ and reduce the Euler system into a single second-order nonlinear elliptic equation for this potential function $\phi$ in the subsonic region. In Section 4, we solve a fixed boundary problem in a bounded, truncated domain of the infinite channel in the new coordinates by carefully making boundary estimates, especially near the corners of the truncated domain, and employing the Hahn–Banach fixed point argument. Then, in Section 5, we extend the solutions of the fixed boundary problem in the bounded, truncated domains to the infinite channel. In Section 6, we formulate an iteration scheme via updating the location of shock front and designing a corresponding map so that the map is a contraction map, which leads to a fixed point that is a solution. Furthermore, in Section 7, we determine the asymptotic behavior of solutions at the infinite exit. In Section 8, we establish the stability of transonic shocks under the small perturbations of both the incoming flows and the nozzle boundaries. In Section 9, we provide the proof of the existence and uniqueness of supersonic solutions in the upstream region, which is required in order to formulate our infinite nozzle problem into a one-phase free boundary problem in Section 2.

## 2. Free boundary problems and main theorem

In this section, we formulate the transonic nozzle problem into a one-phase free boundary problem in the two-dimensional infinite nozzle and state the main theorem.

For concreteness, the nozzle domain can be formulated in the form:

$$\Omega := \{\mathbf{x} \in \mathbb{R}^2\colon\ x_1 > -1,\ \zeta_0(x_1) < x_2 < \zeta_1(x_1)\}, \tag{2.1}$$

where $\zeta_i$, $i = 0, 1$, are functions of $x_1$ to describe the lower and upper walls of the nozzle. Denote the lower and upper boundaries by $\Gamma_i$, i.e.,

$$\Gamma_i := \{\mathbf{x}\colon\ x_2 = \zeta_i(x_1),\ x_1 > -1\}, \quad i = 0, 1. \tag{2.2}$$

We also define:

$$\Omega_1 := \Omega \cap \{-1 < x_1 < 1\}, \tag{2.3}$$

as the entrance part of the flow $U := (\mathbf{u}, p, \rho)$.

We are interested in transonic flows separated by a transonic shock near the $x_2$-axis in the infinite nozzle. Assume that an incoming flow $U_- := (\mathbf{u}_-, p_-, \rho_-)$ is supersonic from the left, defined in the domain $\Omega_1$. Our goal is to seek a shock $S = \{x_1 = s(x_2)\}$ and a downstream subsonic flow $U$ satisfying the Euler equations (1.1) in the domain,

$$\Omega_s := \Omega \cap \{\mathbf{x}\colon\ x_1 > s(x_2)\}, \tag{2.4}$$

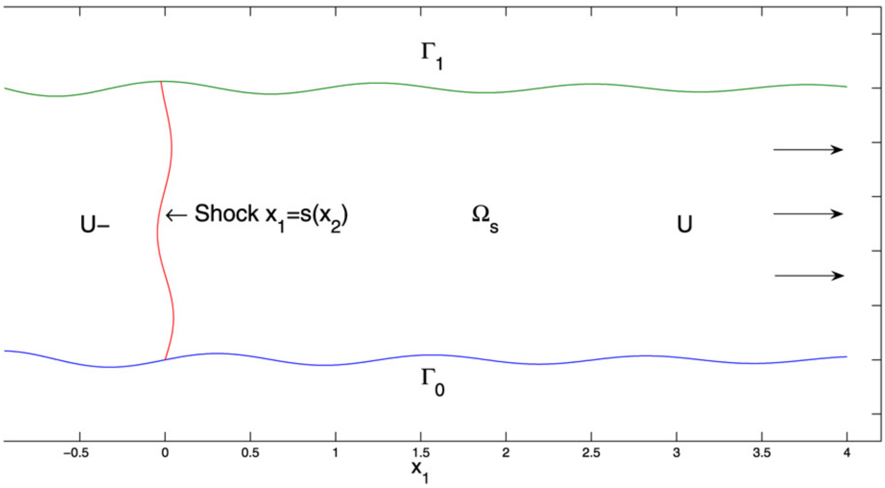


Fig. 1. The infinite transonic nozzle problem.

and the following Rankine–Hugoniot conditions along the shock $x_1 = s(x_2)$:

$$\begin{cases} [\rho u_1] = s'(x_2)[\rho u_2], \\ [\rho u_1^2 + p] = s'(x_2)[\rho u_1 u_2], \\ [\rho u_1 u_2] = s'(x_2)[\rho {u_2}^2 + p], \\ [\rho u_1(E + p/\rho)] = s'(x_2)[\rho u_2(E + p/\rho)], \end{cases} \tag{2.5}$$

where [ ] denotes the jump of the quantity between the two states across the shock front.

Note that $(\mathbf{u}, p, \rho)$, which is smooth in $\Omega_s \cup S$ and $\Omega \setminus \Omega_s$, is a weak solution of the Euler system (1.1) in $\Omega$ if and only if it is a classical solution of (1.1) in $\Omega_s \cup S$ and $\Omega \setminus \Omega_s$, and conditions (2.5) hold on the shock front $S$.

We need to fix one point in the nozzle in order to locate the position of the shock. Without loss of generality, we let $(0, \zeta_0(0))$ be the point for the shock, i.e., $s(\zeta_0(0)) = 0$.

In order to study the infinite nozzle flow of slowly varying cross-sections, we introduce the background solution:

$$U_\pm^0 = \left(u_{1\pm}^0, 0, p_\pm^0, \rho_\pm^0\right),$$

which are two constant states: a supersonic state $U_-^0$ and a subsonic state $U_+^0$ respectively, separated by a steady transonic shock front at $x_1 = 0$. Then

$$U^0 = \begin{cases} U_-^0 = (u_{1-}^0, 0, p_-^0, \rho_-^0) & \text{if } x_1 < 0, \\ U_+^0 = (u_{1+}^0, 0, p_+^0, \rho_+^0) & \text{if } x_1 > 0, \end{cases} \tag{2.6}$$

is a transonic shock solution of (1.1) with (2.5) so that

$$\left|\left(c_\pm^0\right)^2 - \left(u_{1\pm}^0\right)^2\right| > \delta_0, \tag{2.7}$$

for some $\delta_0 > 0$, where $c_\pm^0 = \sqrt{\gamma p_\pm^0/\rho_\pm^0}$. In this case, $s'(x_2) = 0$, and the Rankine–Hugoniot conditions (2.5) become:

$$\begin{cases} [\rho^0 u_1^0] = 0, \\ [\rho^0 (u_1^0)^2 + p^0] = 0, \\ \left[\rho^0 u_1^0 \left(\dfrac{(u_1^0)^2}{2} + \dfrac{\gamma p^0}{(\gamma - 1)\rho^0}\right)\right] = 0. \end{cases} \tag{2.8}$$

Conditions (2.7)–(2.8) yield the entropy condition for the piecewise constant solution:

$$\rho_+^0 > \rho_-^0, \tag{2.9}$$

which implies

$$u_{1+}^0 < u_{1-}^0, \qquad p_+^0 > p_-^0.$$

Since the nozzle has slowly varying cross-sections, the lower and upper boundaries are small perturbations of the straight walls. We assume that, for sufficiently small $\varepsilon > 0$, $\zeta_i = \zeta_i(x_1)$, $i = 1, 2$, satisfy:

$$\|\zeta_i - \mathrm{i}\|_{C^{3,\alpha}(-1,\infty)} \leqslant \varepsilon, \tag{2.10}$$

$$\left|\zeta_i(x_1) - \mathrm{i}\right| \leqslant \frac{\varepsilon}{(1+|x_1|)^\beta}, \tag{2.11}$$

where $1 < \beta < 2$ is a fixed constant. Condition (2.11) indicates that the nozzle asymptotically converges to a uniform nozzle at the infinity with algebraic rate of order $\beta$.

Since our solutions are expected to be near the background solution $U^0$, they will automatically satisfy the entropy condition because of (2.9). In particular, when $U$ satisfies:

$$\left\|U - U_+^0\right\|_{C^\alpha(\Omega_s)} \leqslant C\varepsilon,$$

for some constant $C > 0$, then $U$ stays subsonic in the subsonic region $\Omega_s \subset \Omega$.

Now we set up the transonic nozzle problem. Let $\nu$ be the outer unit normal to $\partial\Omega$.

**Problem I** *(Infinite transonic nozzle problem).* Given a smooth incoming flow close to the uniform supersonic flow $U_0^-$ at the entrance, find a transonic flow $U$ that is supersonic after passing the entrance $\{x_1 = -1\}$ and subsonic in the downstream domain $\Omega_s$, separated by a transonic shock $S := \{x_1 = s(x_2)\}$ with $s(\zeta_0(0)) = 0$ for the following problem of initial-boundary value type in an impermeable nozzle:

$$U|_{x_1=-1} = \tilde{U}_- := (\tilde{\mathbf{u}}_-, \tilde{p}_-, \tilde{\rho}_-)(x_2) \in C^{2,\alpha}(d_0, d_1), \tag{2.12}$$

$$\mathbf{u}\cdot\nu|_{\Gamma_i} = 0, \quad i = 0, 1, \tag{2.13}$$

with the compatibility condition:

$$\tilde{u}_{2-}(d_i) = \zeta_i'(-1)\tilde{u}_{1-}(d_i), \tag{2.14}$$

and the closeness condition to the uniform supersonic flow $U_0^-$ at $x_1 = -1$ and the uniform nozzle for $-1 < x_1 < 1$:

$$\left\|\tilde{U}_- - U_-^0\right\|_{C^{2,\alpha}(d_0,d_1)} + \sum_{i=0,1} \|\zeta_i - \mathrm{i}\|_{C^{3,\alpha}(-1,1)} \leqslant \varepsilon, \tag{2.15}$$

for some small $\varepsilon > 0$, where $d_i = \zeta_i(-1)$, $i = 0, 1$.

In order to formulate this nozzle problem into a free boundary problem, we show the existence and uniqueness of supersonic flows in the upstream region $\Omega_1$. This is achieved via the method of characteristics in Section 9.

**Theorem 2.1.** *There exists $\varepsilon_0 > 0$ such that, when $\varepsilon \in (0, \varepsilon_0)$, there exist a constant $C_0 > 0$ and a unique supersonic solution $U_- = (\mathbf{u}_-, p_-, \rho_-)(x, y) \in C^{2,\alpha}(\Omega_1)$ of problem* (1.1) *and* (2.12)–(2.15) *such that*

$$\left\|U_- - U_-^0\right\|_{C^{2,\alpha}(\Omega_1)} \leqslant C_0\varepsilon. \tag{2.16}$$

With Theorem 2.1, we can reformulate Problem I into the following one-phase free boundary problem.

**Problem II.** Assume that the domain $\Omega$ in (2.1) satisfies condition (2.10)–(2.11). Let $U_-$ be a supersonic solution of (1.1) in the domain $\Omega_1$ satisfying the slip condition (2.13) and the closeness condition to the uniform supersonic flow $U_-^0$:

$$\left\|U_- - U_-^0\right\|_{C^{2,\alpha}(\Omega_1)} \leqslant \varepsilon, \tag{2.17}$$

for some small constant $\varepsilon$. Find a subsonic flow $U(\mathbf{x})$ satisfying the Euler equations (1.1) in the domain $\Omega_s$ defined in (2.4) and the slip condition (2.13), separated by a transonic shock $S = \{x_1 = s(x_2)\}$ satisfying $s(\zeta_0(0)) = 0$ and the Rankine–Hugoniot conditions in (2.5).

At the corners between the shock front $S$ and the boundaries $\Gamma_i$, we expect less regularity for the solution; and this less regularity spreads along the boundaries $\Gamma_i$ so that we will not have the $C^{1,\alpha}$-smoothness for the downstream solution $U$. Instead, we need to use the following weighed Hölder norms (similar to [15]): For any $\mathbf{x}, \mathbf{x}'$ in a two-dimensional domain $E$ and for an open portion $P$ of $\partial E$, define $\delta_{\mathbf{x}} := \operatorname{dist}(\mathbf{x}, P)$ and $\delta_{\mathbf{x},\mathbf{x}'} := \min(\delta_{\mathbf{x}}, \delta_{\mathbf{x}'})$. Let $\alpha \in (0,1)$ and $\sigma \in \mathbb{R}$. We define:

$$\begin{aligned}
&[u]^{(\sigma;P)}_{k,0;E} := \sup_{\mathbf{x}\in E,\, |\mathbf{k}|=k} \big(\delta_{\mathbf{x}}^{\max(k+\sigma,0)} \big|D^{\mathbf{k}} u(\mathbf{x})\big|\big),\\
&[u]^{(\sigma;P)}_{k,\alpha;E} := \sup_{\mathbf{x},\mathbf{x}'\in E,\, \mathbf{x}\neq\mathbf{x}',\, |\mathbf{k}|=k} \left(\delta_{\mathbf{x},\mathbf{x}'}^{\max(k+\alpha+\sigma,0)} \frac{|D^{\mathbf{k}} u(\mathbf{x}) - D^{\mathbf{k}} u(\mathbf{x}')|}{|\mathbf{x}-\mathbf{x}'|^{\alpha}}\right),\\
&\|u\|^{(\sigma;P)}_{k,\alpha;E} := \sum_{i=0}^{k} [u]^{(\sigma;P)}_{i,0;E} + [u]^{(\sigma;P)}_{k,\alpha;E},
\end{aligned} \tag{2.18}$$

where $\mathbf{k} = (k_1, k_2)$, $|\mathbf{k}| = k_1 + k_2$, and $D^{\mathbf{k}} = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2}$. We also use the same notation for one-dimensional domains. In this case, the boundary portion should be understood as the two endpoints.

We define the Banach space $C^{k,\alpha}_{(\sigma;P)}(E)$ by:

$$C^{k,\alpha}_{(\sigma;P)}(E) := \big\{u\colon\ \|u\|^{(\sigma;P)}_{k,\alpha;E} < \infty\big\}. \tag{2.19}$$

We may drop the symbols $P, E$ in the norms within the context if it does not cause ambiguity. For vector-valued functions $U = (u_1, \ldots, u_n)$, we define:

$$\|U\|^{(\sigma;P)}_{k,\alpha;E} := \sum_{i=1}^{n} \|u_i\|^{(\sigma;P)}_{k,\alpha;E}. \tag{2.20}$$

In our problem, the boundary portion is $P = \Gamma_0 \cup \Gamma_1 =: \Gamma_{0,1}$. Now we state our main theorem whose proof is provided in Sections 3–6.

**Theorem 2.2** *(Main theorem). There exists $\varepsilon_0 > 0$ such that, when $\varepsilon \in (0, \varepsilon_0)$, there exist a subsonic solution $U$ and a transonic shock $S$ for Problem* II, *provided that the incoming supersonic flow $U_-$ in $\Omega_1$ satisfies* (2.17). *Furthermore, if the shock $S$ intersects with $\Gamma_0$ and $\Gamma_1$ at points $(x_1^0, x_2^0)$ and $(x_1^1, x_2^1)$ respectively, then*

$$\big\|U - U_+^0\big\|^{(-\alpha;\Gamma_{0,1})}_{1,\alpha;\Omega_s} \leqslant C\varepsilon, \qquad \|s\|^{(-1-\alpha)}_{2,\alpha;(x_2^0,x_2^1)} \leqslant C\varepsilon, \tag{2.21}$$

*where $C$ is a constant depending only on $\alpha$ and $U_-^0$, but independent of $\varepsilon$. The solution $U$ is unique within the class of transonic solutions satisfying* (2.21).

**Remark 2.1.** Combining Theorem 2.2 with Theorem 2.1, we solve the infinite transonic nozzle problem, Problem I.

We also establish the stability and asymptotic behavior of transonic nozzle flows in Sections 7–8.

## 3. Lagrangian coordinates and reduction of the Euler system

To simplify the analysis, we employ the following coordinate transformation of Euler–Lagrange type:

$$\begin{cases} y_1 = x_1,\\ y_2 = \int_{\zeta_0(x_1)}^{x_2} (\rho u_1)(x_1, s)\,\mathrm{d}s, \end{cases} \tag{3.1}$$

under which the original curved streamlines become straight. In the new coordinates $\mathbf{y} = (y_1, y_2)$, we still denote the unknown variables $U(\mathbf{x}(\mathbf{y}))$ by $U(\mathbf{y})$ for simplicity of notation.

The original Euler equations in (1.1) become the following equations in divergence form:

$$\left(\frac{1}{\rho u_1}\right)_{y_1} - \left(\frac{u_2}{u_1}\right)_{y_2} = 0, \tag{3.2}$$

$$\left(u_1 + \frac{p}{\rho u_1}\right)_{y_1} - \left(\frac{p u_2}{u_1}\right)_{y_2} = 0, \tag{3.3}$$

$$(u_2)_{y_1} + p_{y_2} = 0, \tag{3.4}$$

$$\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\right)_{y_1} = 0. \tag{3.5}$$

Let $\mathcal{F}: y_1 = f(y_2)$ be a shock front. Then, from the above equations, we can derive the Rankine–Hugoniot conditions along $\mathcal{F}$:

$$\left[\frac{1}{\rho u_1}\right] = -\left[\frac{u_2}{u_1}\right] f'(y_2), \tag{3.6}$$

$$\left[u_1 + \frac{p}{\rho u_1}\right] = -\left[\frac{p u_2}{u_1}\right] f'(y_2), \tag{3.7}$$

$$[u_2] = [p] f'(y_2), \tag{3.8}$$

$$\left[\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\right] = 0. \tag{3.9}$$

Also the original fixed point $(0, \zeta_0(0))$ on $S$ becomes the origin $(0, 0)$ on the new shock $\mathcal{F}$.

Set:

$$M = \int_{\zeta_0(x_1)}^{\zeta_1(x_1)} (\rho u_1)(x_1, s)\,\mathrm{d}s.$$

It is easy to check that $M$ is a constant independent of $x_1$ from the first equation of (1.1) (conservation of mass). Actually, $M$ is determined by $U_-(-1, x_2)$, the data of incoming flow at the entrance of the nozzle. Also, denote:

$$M_0 = \rho_-^0 u_{1-}^0 = \rho_+^0 u_{1+}^0. \tag{3.10}$$

Then we have:

$$|M - M_0| \leqslant C\varepsilon. \tag{3.11}$$

Under this transformation, the whole domain of the infinite nozzle becomes the infinite channel:

$$R = \{\mathbf{y}:\ y_1 > -1,\ 0 < y_2 < M\}. \tag{3.12}$$

We define:

$$R_1 := R \cap \{y_1 < 1\}, \tag{3.13}$$

$$R_f := R \cap \big\{y_1 > f(y_2)\big\}. \tag{3.14}$$

We also define:

$$\mathcal{B}_0 := R \cap \{y_2 = 0\}, \qquad \mathcal{B}_1 = R \cap \{y_2 = M\}, \tag{3.15}$$

which are the lower and upper boundaries of $R$ respectively. Then we have the slip condition:

$$\left.\frac{u_2}{u_1}\right|_{\mathcal{B}_i} = \zeta_i'(y_1). \tag{3.16}$$

Then Problem II is equivalent to the following problem:

**Problem III.** Let $U_-$ be a supersonic solution satisfying Eqs. (3.2)–(3.5) in the upstream region $R_1$ and the slip condition (3.16). Assume that $U_-$ is a small perturbation of $U_-^0$ with

$$\big\|U_- - U_-^0\big\|_{C^{2,\alpha}(R_-)} \leqslant \varepsilon \tag{3.17}$$

for some small constant $\varepsilon$. Find a subsonic flow $U(\mathbf{y})$ satisfying Eqs. (3.2)–(3.5) in the downstream region $R_f$ and the slip condition (3.16), separated by a transonic shock $\mathcal{F} := R \cap \{y_1 = f(y_2)\}$ satisfying $f(0) = 0$ and the Rankine–Hugoniot conditions (3.6)–(3.9).

Correspondingly, we have the following theorem that is equivalent to Theorem 2.2.

**Theorem 3.1.** *There exists $\varepsilon_0 > 0$ such that, when $\varepsilon \in (0, \varepsilon_0)$, there exist a subsonic solution $U$ and a shock $\mathcal{F} := R \cap \{y_1 = f(y_2)\}$ for Problem* III*, provided that the incoming supersonic flow $U_-$ satisfying* (3.17)*. Furthermore, we have the following estimates*:

$$\|U - U_+^0\|_{1,\alpha;R_f}^{(-\alpha;\mathcal{B}_{0,1})} \leqslant C\varepsilon, \tag{3.18}$$

$$\|f\|_{2,\alpha;(0,M)}^{(-1-\alpha)} \leqslant C\varepsilon, \tag{3.19}$$

*where $C$ is a constant depending only on $\alpha$ and $U^0$, but independent of $\varepsilon$. Within the class of $U$ satisfying* (3.18)*, the solution is unique.*

**Remark 3.1.** Estimate (3.18) guarantees that the coordinate transformation is $C^{3,\alpha}$ smooth in the supersonic region, $C^{1,\alpha}$ in the subsonic region, and Lipschitz across the shock, and the Jacobian is nonsingular, so that we can transform the Lagrangian coordinates back to Euler coordinates, which renders the equivalence of Theorems 3.1 and 2.2.

From now on to the end of Section 6, we focus only on establishing Theorem 3.1 in the **y**-coordinates, which yields Theorem 2.2 in the **x**-coordinates.

According to the coordinate transformation from **x** to **y**, we know that $x_2$ can be solved as a function of **y**. Let $x_2 := \phi(\mathbf{y})$ in the subsonic domain $R_f$ and $x_2 := \phi_-(\mathbf{y})$ in the supersonic domain $R_1$. Given $U_-$, we can find the corresponding $\phi_-$. We now use the function $\phi$ to reduce the original Euler system to an elliptic equation in the subsonic domain.

By the definition of coordinate transformation (3.1), we have:

$$\phi_{y_1} = \frac{u_2}{u_1}, \qquad \phi_{y_2} = \frac{1}{\rho u_1}, \tag{3.20}$$

that is, $\phi(\mathbf{y})$ is the potential function of the vector field $(\frac{u_2}{u_1}, \frac{1}{\rho u_1})$.

Eq. (3.5) implies Bernoulli's law:

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma - 1)\rho} = B(y_2), \tag{3.21}$$

where $B = B(y_2)$ is completely determined by the incoming flow $U_-$ at the entrance $x_1 = -1$.

From Eqs. (3.2)–(3.5), we find:

$$(\gamma \ln \rho - \ln p)_{y_1} = 0,$$

which implies

$$p = A(y_2)\rho^\gamma \qquad \text{in the subsonic domain } R_f. \tag{3.22}$$

The function $A = A(y_2)$ can be determined by the incoming flow $U_-$ and the Rankine–Hugoniot conditions on the shock $\mathcal{F}$, provided that the shock position $y_1 = f(y_2)$ is given. The details of this will be discussed later (see (3.38)–(3.45)).

With Eqs. (3.20) and (3.22), we can rewrite Bernoulli's law into the following form:

$$\frac{\phi_{y_1}^2 + 1}{2\phi_{y_2}^2} + \frac{\gamma}{\gamma - 1} A\rho^{\gamma+1} = B\rho^2. \tag{3.23}$$

In the subsonic region, $|\mathbf{u}| < c := \sqrt{\gamma p/\rho}$. Therefore, Bernoulli's law (3.21) implies:

$$\rho^{\gamma-1} > \frac{2B}{\kappa A}, \tag{3.24}$$

where $\kappa = \frac{\gamma(\gamma+1)}{\gamma-1}$.

Condition (3.24) guarantees that $\rho$ can be solved from (3.23) as a smooth function of $(A, B, \nabla\phi)$.

Assume that we have known $A = A(y_2)$. Then $(\mathbf{u}, p, \rho)$ can be expressed as functions of $\nabla\phi$:

$$\rho = \rho(A, B, \nabla\phi), \quad u_1 = \frac{1}{\rho\phi_{y_2}}, \quad u_2 = \frac{\phi_{y_1}}{\rho\phi_{y_2}}, \quad p = A\rho^\gamma, \tag{3.25}$$

since $B = B(y_2)$ is given by the incoming flow.

We now choose (3.4) to derive a second-order nonlinear elliptic equation for $\phi$ so that the full Euler system is reduced to this equation. Set

$$N^1 = u_2, \qquad N^2 = p - p_+^0. \tag{3.26}$$

Then we have the second-order nonlinear equation for $\phi$:

$$\left(N^1\right)_{y_1} + \left(N^2\right)_{y_2} = 0, \tag{3.27}$$

that is,

$$N^1_{\phi_{y_1}}\phi_{y_1y_1} + \left(N^1_{\phi_{y_2}} + N^2_{\phi_{y_1}}\right)\phi_{y_1y_2} + N^2_{\phi_{y_2}}\phi_{y_2y_2} = 0, \tag{3.28}$$

where $N^i = N^i(A(y_2), B(y_2), \nabla\phi)$, $i = 1, 2$, are given by:

$$N^1(A, B, \nabla\phi) = \frac{\phi_{y_1}}{\phi_{y_2}\rho(A(y_2), B(y_2), \nabla\phi)}, \qquad N^2(A, B, \nabla\phi) = A(y_2)\rho(A(y_2), B(y_2), \nabla\phi)^\gamma - p_+^0. \tag{3.29}$$

We now verify that (3.27), or equivalently (3.28), is uniformly elliptic for $\phi$, provided that $U$ is a small perturbation of $U_+^0$, i.e.,

$$\left\|U - U_+^0\right\|_{C^0(\Omega_f)} \leqslant \delta \tag{3.30}$$

for any small data $\delta \leqslant \delta_0$, where $\delta_0$ will be given later in Lemma 4.1. In the following, the positive constants $\lambda_i$ related to ellipticity will depend only on the background states $U_\pm^0$.

Differentiating (3.23) with respect to $\nabla\phi$, we can calculate $\rho_{\phi_{y_1}}$ and $\rho_{\phi_{y_2}}$ to obtain:

$$\rho_{\phi_{y_1}} = \frac{-\phi_{y_1}}{\phi_{y_2}^2(\kappa A\rho^\gamma - 2B\rho)}, \tag{3.31}$$

$$\rho_{\phi_{y_2}} = \frac{\phi_{y_1}^2 + 1}{\phi_{y_2}^3(\kappa A\rho^\gamma - 2B\rho)}. \tag{3.32}$$

Now we calculate $N^i_{\phi_{y_j}}$, $i, j = 1, 2$. First, we have:

$$N^1_{\phi_{y_1}} = \frac{1}{\phi_{y_2}\rho} - \frac{\phi_{y_1}\rho_{\phi_{y_1}}}{\phi_{y_2}\rho^2} = \frac{\phi_{y_2}^2(\gamma A\rho^{\gamma+1} - \frac{\phi_{y_1}^2+1}{\phi_{y_2}^2}) + \phi_{y_1}^2}{\phi_{y_2}^3\rho^2(\kappa A\rho^\gamma - 2B\rho)}, \tag{3.33}$$

where we used (3.31) and (3.23) to obtain the last equality (3.33). Similarly, using (3.31)–(3.32) and (3.23), we have:

$$N^1_{\phi_{y_2}} = N^2_{\phi_{y_1}} = \frac{-\gamma A\rho^{\gamma-1}\phi_{y_1}}{\phi_{y_2}^2(\kappa A\rho^\gamma - 2B\rho)}, \tag{3.34}$$

$$N^2_{\phi_{y_2}} = \frac{\gamma A\rho^{\gamma-1}(\phi_{y_1}^2 + 1)}{\phi_{y_2}^3(\kappa A\rho^\gamma - 2B\rho)}. \tag{3.35}$$

By subsonicity of $U_+^0$ and the small perturbation assumption (3.30), we have:

$$\kappa A\rho^\gamma - 2B\rho > \lambda_1 > 0.$$

Hence, we know $N^2_{\phi_{y_2}} > \lambda_2 > 0$. In the subsonic domain, we have:

$$|\mathbf{u}|^2 = \frac{\phi_{y_1}^2 + 1}{\phi_{y_2}^2\rho^2} < c^2 - \lambda_3 = \gamma A\rho^{\gamma-1} - \lambda_3.$$

Therefore, (3.33) implies $N^1_{\phi_{y_1}} > \lambda_4 > 0$.

Then we have:

$$N^1_{\phi_{y_1}} N^2_{\phi_{y_2}} - N^1_{\phi_{y_2}} N^2_{\phi_{y_1}} = \frac{\gamma A\rho^{\gamma-1}(\gamma A\rho^{\gamma+1} - \frac{\phi^2_{y_1}+1}{\phi^2_{y_2}})}{\phi^4_{y_2}\rho^2(\kappa A\rho^\gamma - 2B\rho)^2} > \lambda_5 > 0, \tag{3.36}$$

which implies that the second-order nonlinear equation (3.27), equivalently (3.28), is uniformly elliptic.

Next we derive the boundary conditions for Eq. (3.27). Among the four Rankine–Hugoniot conditions (3.6)–(3.9), condition (3.6) is equivalent to the continuity of $\phi$ across the shock $\mathcal{F}$. That is,

$$\phi = \phi_- \quad \text{on the shock } \mathcal{F}. \tag{3.37}$$

Condition (3.8) will be used to locate the shock front later, and condition (3.9) is already used to compute $B$. We need all the four conditions (3.6)–(3.9) to determine the function $A = A(y_2)$.

To compute $A$, we first fix the shock front $\mathcal{F}$. Then, from (3.8), we can determine $u_2$:

$$u_2 = u_{2-} + [p]f'(y_2). \tag{3.38}$$

Let $W = (u_1, p, \rho)$. Eliminating $u_2$ in (3.6)–(3.7) and (3.9) by (3.38) leads to:

$$\left[\frac{1}{\rho u_1}\right] + \tilde{G}_1(U_-, W, f') =: G_1(U_-, W, f') = 0, \tag{3.39}$$

$$\left[u_1 + \frac{p}{\rho u_1}\right] + \tilde{G}_2(U_-, W, f') =: G_2(U_-, W, f') = 0, \tag{3.40}$$

$$\left[\frac{1}{2}u_1^2 + \frac{\gamma p}{(\gamma-1)\rho}\right] + \tilde{G}_3(U_-, W, f') =: G_3(U_-, W, f') = 0, \tag{3.41}$$

along the shock $\mathcal{F}$, where

$$\tilde{G}_i(U_-, W, f') = O\big(|u_{2-}| + |f'|^2\big).$$

Let $\mathbf{G} = (G_1, G_2, G_3)^\top$. We compute $\nabla_W \mathbf{G} \equiv (\partial_{u_1}\mathbf{G}, \partial_p\mathbf{G}, \partial_\rho\mathbf{G})$:

$$\nabla_W \mathbf{G}|_{\{U_-=U^0_-, f'=0\}} = \begin{pmatrix} -\frac{1}{\rho u_1{}^2} & 0 & -\frac{1}{\rho^2 u_1} \\ 1 - \frac{p}{\rho u_1{}^2} & \frac{1}{\rho u_1} & -\frac{p}{\rho^2 u_1} \\ u_1 & \frac{\gamma}{(\gamma-1)\rho} & -\frac{\gamma p}{(\gamma-1)\rho^2} \end{pmatrix}. \tag{3.42}$$

It is easy to check that

$$\det(\nabla_W \mathbf{G}) = -\frac{c^2 - u_1^2}{(\gamma-1)\rho^3 u_1^3} < -\lambda_6 < 0 \tag{3.43}$$

for some constant $\lambda_6$ depending only on $U^0$.

Therefore, by the Implicit Function Theorem, we can solve Eqs. (3.39)–(3.41) for $W$ along the shock $y_1 = f(y_2)$. Then we define:

$$A(y_2) = \left(\frac{p}{\rho^\gamma}\right)\big(f(y_2), y_2\big). \tag{3.44}$$

Let $A^0 = \frac{p^0_+}{(\rho^0_+)^\gamma}$ and $W^0_+ = (u^0_{1+}, \rho^0_+, p^0_+)$. From (3.39)–(3.41), we know:

$$\mathbf{G}\big(U^0_-, W^0_+, 0\big) = \mathbf{0}.$$

Hence, by the Taylor expansion, we conclude that $A$ is a small perturbation of $A^0$:

$$A - A^0 = O\big(\big|U_-\big(f(y_2), y_2\big) - U^0_-\big| + \big|f'(y_2)\big|^2\big). \tag{3.45}$$

The conditions for $\phi$ on the lower and upper boundaries $\mathcal{B}_{0,1}$ are:

$$\phi|_{\mathcal{B}_i} = \zeta_i \quad \text{for } i = 0, 1. \tag{3.46}$$

This is equivalent to the slip condition (3.16).

Finally, for the fixed shock $\mathcal{F}$, we heuristically determine the *asymptotic behavior* of $\phi$ as $y_1 \to \infty$. Let $\phi \to \phi_\infty(y_2)$ as $y_1 \to \infty$. From Eq. (3.27), $N^2 + p_+^0 = p$ approaches to a constant, denoted by $p_\infty$. Then

$$\rho \to \left(\frac{p_\infty}{A}\right)^{1/\gamma} \quad \text{as } y_1 \to \infty.$$

By Bernoulli's law (3.23), we have:

$$\frac{1}{2\phi'_\infty(y_2)^2} + \frac{\gamma}{\gamma-1} A \left(\frac{p_\infty}{A}\right)^{(\gamma+1)/\gamma} = B \left(\frac{p_\infty}{A}\right)^{2/\gamma}. \tag{3.47}$$

We can solve $\phi'_\infty(y_2)$ from the above equation to obtain:

$$\phi'_\infty(y_2) = \left(2B\left(\frac{p_\infty}{A}\right)^{2/\gamma} - \frac{2\gamma}{\gamma-1} A \left(\frac{p_\infty}{A}\right)^{(\gamma+1)/\gamma}\right)^{-1/2}. \tag{3.48}$$

We also have the following boundary condition for $\phi_\infty(y_2)$:

$$\phi_\infty(0) = 0, \qquad \phi_\infty(M) = 1. \tag{3.49}$$

Integrating (3.48) and using condition (3.49), we can uniquely determine the constant $p_\infty$ and the function $\phi = \phi_\infty(y_2)$. This can be seen as follows.

Define a function $h$ by:

$$h(A, B, p_\infty) := 2B\left(\frac{p_\infty}{A}\right)^{2/\gamma} - \frac{2\gamma}{\gamma-1} A \left(\frac{p_\infty}{A}\right)^{(\gamma+1)/\gamma}.$$

Define a functional $H : C([0, M]) \times C([0, M]) \times \mathbb{R}^2 \to \mathbb{R}$ by:

$$H(A, B, M, p_\infty) := \int_0^M \frac{\mathrm{d}y_2}{\sqrt{h(A(y_2), B(y_2), p_\infty)}}.$$

Therefore, if $\phi_\infty$ and $p_\infty$ satisfy (3.48) and (3.49) for a given $(A, B, M)$, then we have:

$$H(A, B, M, p_\infty) = \phi_\infty(M) = 1. \tag{3.50}$$

We now prove Eq. (3.50) is uniquely solvable for $p_\infty$ by showing $\frac{\partial H}{\partial p_\infty}$ is strictly positive.

Actually, it is not hard to check that

$$\frac{\partial H}{\partial p_\infty}\big(A^0, B^0, M_0, p_+^0\big) = \frac{\rho_+^0((c_+^0)^2 - (u_{1+}^0)^2)}{M_0^2 (c_+^0)^2} > \lambda_7 > 0,$$

where $\lambda_7$ depends only on $U_\pm^0$, and $B^0 = \frac{1}{2}(u_{1+}^0)^2 + \frac{\gamma p_+^0}{(\gamma-1)\rho_+^0}$. Computing $\frac{\partial H}{\partial p_\infty}$ explicitly, we see that $\frac{\partial H}{\partial p_\infty}$ is continuous with respect to $(A, B, M, p_\infty) \in C([0, M]) \times C([0, M]) \times \mathbb{R}^2$. Hence, there exists a small constant $\delta > 0$, depending on $U_\pm^0$, such that

$$\frac{\partial H}{\partial p_\infty}(A, B, M, p_\infty) > \frac{\lambda_7}{2},$$

provided that

$$\big\|\big(A - A^0, B - B^0\big)\big\|_C + |M - M_0| + \big|p_\infty - p_+^0\big| \leqslant \delta. \tag{3.51}$$

Obviously, we have a trivial solution for (3.50):

$$H\big(A^0, B^0, M_0, p_+^0\big) = 1.$$

This implies that $p_\infty$ is uniquely determined when (3.51) is satisfied.

**Remark 3.2.** After we obtain estimate (5.3) later in Lemma 5.1, the function $\phi_\infty = \phi_\infty(y_2)$ derived above will be assured to be the asymptotic state.

## 4. Fixed boundary problems in finite domains

In this section we solve a fixed boundary problem in the truncated domain of $R$:

$$R^Q := R \cap \{\mathbf{y}\colon\ y_1 < Q\}, \tag{4.1}$$

where $Q$ is a constant greater than 1. Let

$$R_f^Q := R \cap \{\mathbf{y}\colon\ f(y_2) < y_1 < Q\}, \qquad \mathcal{E} := \{y_1 = Q\} \cap R. \tag{4.2}$$

We prescribe the condition at the exit $\mathcal{E}$:

$$\phi|_{\mathcal{E}} = h_Q, \tag{4.3}$$

with

$$h_Q = \zeta_0(Q) + \big(\zeta_1(Q) - \zeta_0(Q)\big)\phi_\infty(y_2). \tag{4.4}$$

Using (3.49), it is easy to see the compatibility of the boundary conditions (3.46) and (4.3):

$$h_Q(0) = \zeta_0(Q), \qquad h_Q(M) = \zeta_1(Q). \tag{4.5}$$

Now we formulate our fixed boundary problem in the finite domain $R_f^Q$.

**Problem IV.** Given an incoming flow $\phi_-$ with:

$$\left\| \phi_- - \frac{y_2}{M_0} \right\|_{2,\alpha;R_1} \leqslant \varepsilon,$$

and the fixed shock front $\mathcal{F} := \{y_1 = f(y_2)\}$, find a solution $\phi$ of the following boundary value problem:

$$\sum_{i=1,2} \big(N^i\big(A(y_2), B(y_2), \nabla\phi\big)\big)_{y_i} = 0 \quad \text{in } R_f^Q, \tag{4.6}$$

$$\phi|_{\mathcal{F}} = \phi_-|_{\mathcal{F}}, \quad \phi|_{\mathcal{B}_{0,1}} = \zeta_{0,1}, \quad \phi|_{\mathcal{E}} = h_Q. \tag{4.7}$$

Then we have the following lemma for Problem IV.

**Lemma 4.1.** *There exist a constant $C$, a small constant $\delta_0 > 0$, and a constant $\varepsilon_0(\delta) > 0$ for each $0 < \delta < \delta_0$ with the following properties: If $Q > 1$, $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0(\delta))$, and a shock function $f = f(y_2)$ satisfies:*

$$\|f\|_{2,\alpha;(0,M)}^{(-1-\alpha)} \leqslant \delta,$$

*then there is a solution $\phi$ of Problem* IV *satisfying:*

$$\|\phi - \phi_\infty\|_{2,\alpha;R_f^Q}^{(-1-\alpha;\mathcal{B}_{0,1})} \leqslant C\big(\varepsilon + \delta^2\big), \tag{4.8}$$

*where the norms used above are defined in* (2.18)*. Moreover, the solution is unique in the class $\{\phi\colon\ \|\phi - \phi_\infty\|_{2,\alpha;R_f^Q}^{(-1-\alpha;\mathcal{B}_{0,1})} \leqslant \delta\}$.*

Before we prove Lemma 4.1, we need a technical lemma for the elliptic estimates. Since we deal only with the finite domain $R_f^Q$ in this section, we simplify the notation by omitting the domain $R_f^Q$ and the boundaries $\mathcal{B}_{0,1}$ for the weights in the weighted norms.

Consider the boundary value problem for the elliptic equation,

$$\sum_{i,j=1,2} \big(a_{ij}(\mathbf{y})\varphi_{y_i}\big)_{y_j} = \sum_{i=1,2} \big(b_i(\mathbf{y})\big)_{y_i} \quad \text{in } R_f^Q, \tag{4.9}$$

with the boundary conditions:

$$\varphi|_{y_1 = f(y_2)} = g_S(y_2), \quad \varphi|_{\mathcal{B}_{0,1}} = g_{0,1}(y_1), \quad \varphi|_{\mathcal{E}} = g_{\mathcal{E}}(y_2). \tag{4.10}$$

Denote the intersection points of the shock $\mathcal{F}$ and the upper boundary $\mathcal{B}_1$ by $S_1 = (y_1^1, M)$. Also, the intersection points of the exit $\mathcal{E}$ with $\mathcal{B}_{0,1}$ are denoted by $E_0$ and $E_1$. We require the compatibility condition for the boundary data (4.10) so that $\varphi$ is continuous on the boundary of $R_f^Q$:

$$g_0(0) = g_S(0), \quad g_1\big(y_1^1\big) = g_S(M), \quad g_0(Q) = g_{\mathcal{E}}(0), \quad g_1(Q) = g_{\mathcal{E}}(M). \tag{4.11}$$

We also assume:

$$g_{0,1} \in C^{1,\alpha}(\mathcal{B}_{0,1}), \tag{4.12}$$

$$g_S, g_{\mathcal{E}} \in C^{(-1-\alpha)}_{2,\alpha;(0,M)}. \tag{4.13}$$

The coefficients $a_{ij}$ and $b_i$ satisfy:

$$\sum_{i,j=1,2} \|a_{ij}\|^{(-\alpha)}_{1,\alpha} + \sum_{i=1,2} \|b_i\|^{(-\alpha)}_{1,\alpha} \leqslant \Lambda, \tag{4.14}$$

$$\|a_{ij} - e_i\delta_{ij}\|^{(-\alpha)}_{1,\alpha} \leqslant \bar{\delta}, \quad i, j = 1, 2, \tag{4.15}$$

$$\sum_{i,j=1,2} a_{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2, \tag{4.16}$$

where $\xi = (\xi_1, \xi_2)$ is an arbitrary vector; $\bar{\delta}$ is a fixed small constant depending on $U^0$; $\lambda, \Lambda, e_i$ are fixed positive constants, depending on $U^0$; and $\delta_{ij} = 1$ for $i = j$, and 0 otherwise.

**Lemma 4.2.** *There is a unique solution* $\varphi \in C^{(-1-\alpha;\mathcal{B}_{0,1})}_{2,\alpha;R_f^Q}$ *for the boundary value problem* (4.9)–(4.10) *satisfying* (4.12)–(4.16). *Furthermore, the solution* $\varphi$ *satisfies*:

$$\|\varphi\|^{(-1-\alpha)}_{2,\alpha} \leqslant C\left(\sum_{i=0,1} \|g_i\|_{1,\alpha;\mathcal{B}_i} + \sum_{i=S,\mathcal{E}} \|g_i\|^{(-1-\alpha)}_{2,\alpha;(0,M)} + \sum_{i=1,2} \|b_i\|^{(-\alpha)}_{1,\alpha}\right), \tag{4.17}$$

*where* $C$ *is a constant depending only on* $\lambda$, $\Lambda$, $\bar{\delta}$, *and* $e_i$, *but independent of the length of the nozzle* $Q$.

**Proof.** The boundary value problem (4.9)–(4.10) admits a unique solution in $C^0(\overline{R_f^Q}) \cap C^{2,\alpha}(R_f^Q)$, which is a classical result (cf. [16], Chapter 8). To obtain the estimates in the weighted norm $C^{2,\alpha}_{(-1-\alpha)}$ independent of the domain, a detailed analysis is required. Our estimates below are motivated by the techniques in [17] and [20–22]

By assumption (4.15), the elliptic operator,

$$L = \sum_{i,j=1,2} \partial_{y_i}(a_{ij}\partial_{y_j}),$$

is a small perturbation of the operator,

$$L_0 = e_1\partial^2_{y_1} + e_2\partial^2_{y_2}.$$

Without loss of generality, we may assume:

$$e_1 = e_2 = 1.$$

Otherwise, by the transformation $(\tilde{y}_1, \tilde{y}_2) = (\frac{y_1}{\sqrt{e_1}}, \frac{y_2}{\sqrt{e_2}})$, we can change $L_0$ into the Laplace operator.

We first need to obtain a $C^0$-estimate of $\varphi$, which depends only on the boundary data and nonhomogeneous terms, but independent of the domain. To achieve this, we use the following comparison function:

$$v_0 = c_0\big(1 + {y_2}^\alpha + (M - y_2)^\alpha\big),$$

where

$$c_0 = \tilde{c}_0\left(G_b + \sum_{i=1,2} \|b_i\|^{(-\alpha)}_{1,\alpha}\right), \quad G_b = \sum_{i=0,1} \|g_i\|_{1,\alpha;\mathcal{B}_i} + \sum_{i=S,\mathcal{E}} \|g_i\|^{(-1-\alpha)}_{2,\alpha;(0,M)}.$$

Then, using (4.15), we have:

$$\begin{aligned}
Lv_0 &= c_0\alpha(\alpha-1)\big({y_2}^{\alpha-2}+(M-y_2)^{\alpha-2}\big)+c_0(a_{2i}-\delta_{2i})_{y_i}\alpha\big({y_2}^{\alpha-1}-(M-y_2)^{\alpha-1}\big)\\
&\leqslant -\frac{1}{2}\alpha(1-\alpha)c_0\big({y_2}^{\alpha-2}+(M-y_2)^{\alpha-2}\big)\\
&\leqslant -\frac{1}{2M}\alpha(1-\alpha)\tilde{c}_0\big({y_2}^{\alpha-1}+(M-y_2)^{\alpha-1}\big)\sum_{i=1,2}\|b_i\|_{1,\alpha}^{(-\alpha)}\\
&\leqslant \sum_{i=1,2}(b_i)_{y_i}=L\varphi,
\end{aligned}$$

for sufficiently large $\tilde{c}_0$.

On the boundary, $v_0 \geqslant \varphi$ by definition of $v_0$. Hence, by the Comparison Principle, we conclude:

$$\|\varphi\|_{0,0}\leqslant v_0\leqslant C\bigg(G_b+\sum_{i=1,2}\|b_i\|_{1,\alpha}^{(-\alpha)}\bigg), \tag{4.18}$$

where $C$ is independent of $Q$.

To obtain a better maximum bound, we construct a comparison function $v$. We focus on the behavior of $\varphi$ near the origin. The other corner points can be treated in the same way. Let $\tau=(1-\alpha)/5$. Define the function $v(r,\theta)$ in polar coordinates:

$$v(r,\theta)=c_1r^{1+\alpha}\sin\big(\tau+(1+\alpha+\tau)\theta\big)-c_2{y_2}^{1+\alpha}=:c_1v_1-c_2v_2, \tag{4.19}$$

where the constants $c_1$ and $c_2$ will be determined later. A simple computation shows that

$$\Delta v_1=\big((1+\alpha)^2-(1+\alpha+\tau)^2\big)r^{\alpha-1}\sin\big(\tau+(1+\alpha+\tau)\theta\big)\leqslant -2r^{\alpha-1}\tau\sin\tau.$$

Together with

$$\Delta v_2=\alpha(1+\alpha){y_2}^{\alpha-1},$$

we obtain:

$$\Delta v\leqslant -c_2\alpha(1+\alpha){y_2}^{\alpha-1}<0. \tag{4.20}$$

At the corner $S_0=(0,0)$, we assume that

$$\varphi(0,0)=0,\qquad g_0(0)=g_S(0)=g_0'(0)=g_S'(0)=0.$$

Otherwise, we can replace $\varphi$ with $\varphi-g_0(0)-g_0'(0)y_1-(g_S'(0)-g_0'(0)f'(0))y_2$. Hence, the following inequalities hold:

$$\begin{aligned}
\big|g_0(y_1)\big| &\leqslant \|g_0\|_{1,\alpha;\mathcal{B}_0}y_1^{1+\alpha}\leqslant G_br^{1+\alpha},\\
\big|g_S(y_2)\big| &\leqslant \|g_S\|_{1,\alpha;(0,M)}\,{y_2}^{1+\alpha}\leqslant G_br^{1+\alpha}.
\end{aligned}$$

We fix a radius $r_0$ and let $B_{r_0}^+=B_{r_0}(0)\cap R_f^Q$. The estimate for $\Delta v$ with the conditions in (4.15) yield that, for sufficiently large $c_2$ and small $\bar{\delta}$ in $B_{r_0}^+$,

$$\begin{aligned}
Lv &= \Delta v+(L-\Delta)v\\
&\leqslant -c_2\alpha(1+\alpha){y_2}^{\alpha-1}+\sum_{i,j=1,2}(a_{ij}-\delta_{ij})v_{y_iy_j}+\sum_{i,j=1,2}(a_{ij})_{x_i}v_{x_j}\\
&\leqslant -c_2\alpha(1+\alpha){y_2}^{\alpha-1}+c_2\bar{\delta}{y_2}^{\alpha-1}+c_2\bar{\delta}{y_2}^{\alpha-1}r^{\alpha}\\
&\leqslant \sum_{i=1,2}(b_i)_{y_i}.
\end{aligned}$$

Let $c_1:=\tilde{c}_1(G_b+\sum_{i=1,2}\|b_i\|_{1,\alpha}^{(-\alpha)})$. We analyze the value of $v$ on the boundary near the corner $S_0=(0,0)$ to find:

$$v|_{y_2=0}=\tilde{c}_1\bigg(G_b+\sum_{i=1,2}\|b_i\|_{1,\alpha}^{(-\alpha)}\bigg){y_1}^{\alpha+1}\sin\tau\geqslant g_0,$$

for sufficiently large $\tilde{c}_1$. Also, we have:

$$v|_S \geqslant \frac{1}{2}\tilde{c}_1 G_b r^{1+\alpha} \sin\tau \geqslant g_S,$$

for large $\tilde{c}_1$. Observing (4.18), we have:

$$v|_{r=r_0} \geqslant \frac{1}{2}\tilde{c}_1\Bigg(G_b + \sum_{i=1,2} \|b_i\|_{1,\alpha}^{(-\alpha)}\Bigg){r_0}^{1+\alpha} \geqslant \varphi|_{r=r_0}.$$

Therefore, we conclude:

$$v|_{\partial B_{r_0}^+} \geqslant \varphi|_{\partial B_{r_0}^+}.$$

By the Comparison Principle, we obtain:

$$\varphi \leqslant v \leqslant C\Bigg(G_b + \sum_{i=1,2} \|b_i\|_{1,\alpha}^{(-\alpha)}\Bigg) r^{1+\alpha} \quad \text{in } B_{r_0}^+.$$

In the same way, we can show that $\varphi \geqslant -v$ in $B_{r_0}^+$.

Therefore, we have:

$$|\varphi| \leqslant C\Bigg(G_b + \sum_{i=1,2} |b_i|_{1,\alpha}^{(-\alpha)}\Bigg) r^{1+\alpha}. \tag{4.21}$$

With estimate (4.21), we can use the scaling technique to obtain the $C^{1,\alpha}$-estimate up to the corner. More precisely, for any point $P_0 \in B_{r_0/2}^+$ with polar coordinates $(d_0, \theta_0)$, we consider two cases for different values of $\theta_0$.

*Case* 1. $\theta_0 \in [\pi/6, \pi/3]$. Let $B_1 = B_{d_0/6}(P_0)$ and $B_2 = B_{d_0/3}(P_0)$. Then $B_1 \subset B_2 \subset B_{r_0}^+$. By the Schauder interior estimates, we have:

$$\|\varphi\|_{1,\alpha;B_2}^{(0)} \leqslant C\|\varphi\|_{0,0;B_2},$$

where $C$ is a constant independent of $d_0$, and the weight of the norm is up to $\partial B_2$. Therefore, by (4.21), we conclude:

$$\|\varphi\|_{1,\alpha;B_1} \leqslant C\Bigg(G_b + \sum_{i=1,2} |b_i|_{1,\alpha}^{(-\alpha)}\Bigg). \tag{4.22}$$

*Case* 2. $\theta_0 > \pi/3$ or $\theta_0 < \pi/6$. Let $B_3 = R_f^Q \cap B_{2d_0/3}(P_0)$. By the Schauder boundary estimate, we have:

$$\|\varphi\|_{1,\alpha;B_3}^{(0)} \leqslant C\Bigg(G_b + \|\varphi\|_{0,0} + \sum_{i=1,2} \|b_i\|_{0,\alpha}\Bigg) \leqslant C\Bigg(G_b + \sum_{i=1,2} \|b_i\|_{1,\alpha}^{(-\alpha)}\Bigg).$$

Combining Case 1 with Case 2 yields the following corner estimate:

$$\|\varphi\|_{1,\alpha;B_{r_0/2}^+} \leqslant C\Bigg(G_b + \sum_{i=1,2} \|b_i\|_{1,\alpha}^{(-\alpha)}\Bigg). \tag{4.23}$$

The other three corners can be treated in the same way. Away from the four corners, we have the standard Schauder boundary and interior estimates. We conclude the $C^{1,\alpha}$-estimate:

$$\|\varphi\|_{1,\alpha;R_f^Q} \leqslant C\Bigg(G_b + \sum_{i=1,2} |b_i|_{1,\alpha;R_f^Q}^{(-\alpha;\mathcal{B}_{0,1})}\Bigg). \tag{4.24}$$

We differentiate Eq. (4.9) with respect to $y_k$ for $k = 1, 2$ and obtain:

$$\sum_{i,j=1,2} \big(a_{ij}(\varphi_{y_k})_{y_i}\big)_{y_j} = \sum_{i,j=1,2} \big((b_i)_{y_k} - (a_{ij})_{y_k}\varphi_{y_j}\big)_{y_i} =: \tilde{b}_k. \tag{4.25}$$

Let $P_0 = (y_1^0, y_2^0) \in R_f^Q$. We consider two cases. In the first case, $P_0$ is away from the boundary $\mathcal{F}$, and we use the Schauder interior estimates. In the second case, for $P_0$ close to the shock $\mathcal{F}$, we use the Schauder boundary estimate with oblique boundary condition.

*Case* 1. For the polar angle $\theta_0 < \pi/3$, let $B_1 = B_{y_2^0/4}(P_0)$ and $B_2 = B_{y_2^0/2}(P_0)$. Denote $\varphi_{y_k}$ by $u$. Then Eq. (4.25) is an elliptic equation for $u$. By the Schauder estimate and using (4.15) and (4.24), we have:

$$\begin{aligned}\|u\|_{1,\alpha;B_1} &\leqslant C\Big(\|u\|_{0,0;B_2} + \sum_{i,j,k=1,2}\big\|(b_i)_{y_k} - (a_{ij})_{y_k}\varphi_{y_j}\big\|_{0,\alpha;B_2}\Big)\\ &\leqslant C\Big(G_b + \sum_{i,j=1,2}\big(\|b_i\|_{1,\alpha}^{(-\alpha)} + \|b_i\|_{1,\alpha;B_2} + \|a_{ij}\|_{1,\alpha;B_2}\|\varphi_{y_j}\|_{0,\alpha;B_2}\big)\Big)\\ &\leqslant C|y_2^0|^{-1}\Big(G_b + \sum_{i=1,2}\|b_i\|_{1,\alpha}^{(-\alpha)}\Big).\end{aligned} \tag{4.26}$$

*Case* 2. $\theta_0 \geqslant \pi/3$. This is for the point $P_0$ close to the shock $\mathcal{F}$. Let $u = \varphi_{y_1}$. We have the elliptic equation (4.25) for $u$ when $k = 1$.

We define $B_1 := B_{5y_1^0/4}(P_0) \cap R_f^Q$ and $B_2 := B_{3y_1^0/2}(P_0) \cap R_f^Q$. Then $B_1 \subset B_2 \subset R_f^Q$. Now we need to derive an oblique boundary condition for $u$.

Note that $\varphi(f(y_2), y_2) = g_S(y_2)$. Let $D_\tau := f'(y_2)\partial_{y_1} + \partial_{y_2}$. Therefore, we have:

$$D_\tau^2\varphi = (f')^2\varphi_{y_1y_1} + 2f'\varphi_{y_1y_2} + \varphi_{y_2y_2} + f''\varphi_{y_1} = g_S''. \tag{4.27}$$

Eq. (4.9) yields:

$$\varphi_{y_2y_2} = (\bar g - a_{11}\varphi_{1y_1} - 2a_{12}\varphi_{y_1y_2})/a_{22},$$

where $\bar g = \sum_{i,j=1,2}((b_i)_{y_i} - (a_{ij})_{y_j}\varphi_{y_i})$. Eliminating $\varphi_{y_2y_2}$ in (4.27) gives:

$$\nu_1 u_{y_1} + \nu_2 u_{y_2} = \nu_3 \quad \text{on the shock } S, \tag{4.28}$$

where

$$\begin{aligned}(\nu_1,\nu_2) &= \Big((f')^2 - \frac{a_{11}}{a_{22}}, 2f' - \frac{2a_{12}}{a_{22}}\Big) \in C^\alpha\big(R_f^Q;\mathbb{R}^2\big),\\ \nu_3 &= g_S'' - f''\varphi_{y_1} - \frac{\bar g}{a_{22}} \in C^{0,\alpha}_{(1;\mathcal{B}_{0,1})}\big(R_f^Q\big).\end{aligned}$$

Since $\nu_1 < -\frac{e_1}{2e_2}$, condition (4.28) is an oblique boundary condition. The boundary estimate in $B_2$ results in

$$\|u\|_{1,\alpha;B_1} \leqslant C\big(\|u\|_{0,0;B_2} + \|\nu_3\|_{0,\alpha;B_2} + \big\|\tilde b\big\|_{0,\alpha;B_2}\big) \leqslant C\big|y_2^0\big|^{-1}\Big(G_b + \sum_{i=1,2}\|b_i\|_{1,\alpha}^{(-\alpha)}\Big),$$

where $\tilde b$ is defined in (4.25). Once we obtain the above estimate for $u = \varphi_{y_1}$, by Eq. (4.9) itself, we obtain the same estimate for $\varphi_{y_2}$. Together with estimate (4.26), we obtain estimate (4.17).

The uniqueness follows from estimate (4.17) and the linearity of the problem. This completes the proof of Lemma 4.2. □

Now we come back to the proof of Lemma 4.1.

**Proof of Lemma 4.1.** By the method for finding $\phi_\infty = \phi_\infty(y_2)$ in Section 3 (cf. (3.47)), we know that $\phi_\infty = \phi_\infty(y_2)$ satisfies Eq. (4.6), that is,

$$\sum_{i=1,2}\big(N^i\big(A(y_2), B(y_2), 0, \phi_\infty'(y_2)\big)\big)_{y_i} = 0. \tag{4.29}$$

Taking the difference of (4.6) and (4.29), we have:

$$\sum_{i,j=1,2}\big(a_{ij}^\phi(\phi - \phi_\infty)_{y_i}\big)_{y_j} = 0, \tag{4.30}$$

where

$$a_{ij}^{\phi}=\int_0^1 N^i_{\phi_{y_j}}\big(A,B,\nabla\big(\phi_\infty+s(\phi-\phi_\infty)\big)\big)\,\mathrm{d}s,\quad i,j=1,2. \tag{4.31}$$

To solve the nonlinear equation (4.6), we construct a map $T:\Sigma\to\Sigma$, where

$$\Sigma=\big\{\phi:\ \|\phi-\phi_\infty\|^{(-1-\alpha)}_{2,\alpha;R_f^Q}\leqslant\delta\big\}.$$

By proving that $T$ is a contraction map, we establish the existence and uniqueness of the solution of (4.6).

We first define the map $T$. For any given $\phi\in\Sigma$, we solve the linearized equation:

$$\sum_{i,j=1,2}\big(a_{ij}^{\phi}\big(\tilde\phi-\phi_\infty\big)_{y_i}\big)_{y_j}=0 \tag{4.32}$$

with boundary condition (3.46), (4.3), and (3.37). We define the solution $\tilde\phi:=T\phi$. Now we need to prove that

(i) $T$ is well-defined and maps $\Sigma$ to itself;
(ii) $T$ is a contraction map.

For a given $\phi$, let $a_{ij}:=a_{ij}^{\phi}$. We know that $\tilde\phi-\phi_\infty$ satisfies the linear equation (4.9) in Lemma 4.2. In order to apply Lemma 4.2, we need to verify conditions (4.12)–(4.16). Since $\phi\in\Sigma$, we have:

$$\|a_{ij}-e_i\delta_{ij}\|^{(-\alpha)}_{1,\alpha}\leqslant C_1\delta,$$

where

$$e_1=u^0_{1+},\qquad e_2=\frac{(c^0_+)^2\rho^0_+(u^0_{1+})^3}{(c^0_+)^2-(u^0_{1+})^2}$$

are obtained through (3.33) and (3.35) respectively, by plugging the background state $U^0_+$. For $\delta<\bar\delta/C_1$, condition (4.15) in Lemma 4.2 is satisfied. Conditions (4.14) and (4.16) can be also verified. By Lemma 4.2, we can uniquely solve $\tilde\phi$ and obtain the following estimate:

$$\begin{aligned}\big\|\tilde\phi-\phi_\infty\big\|^{(-1-\alpha)}_{2,\alpha}&\leqslant C\Big(\|\phi_--\phi_\infty\|^{(-1-\alpha)}_{2,\alpha}+\sum_{i=0,1}\|\zeta_i-i\|_{1,\alpha}+\|h_Q-\phi_\infty\|^{(-1-\alpha)}_{2,\alpha}\Big)\\&\leqslant C\big(\varepsilon+\delta^2\big).\end{aligned}$$

In the last inequality, we use the fact that

$$\phi_\infty-\frac{y_2}{M}=\mathrm{O}\big(\big|U_--U^0_-\big|+|f'|^2\big),$$

which is obtained by (3.45), (3.48), and (3.49). As long as $\delta<\delta_0:=\min(\frac{\bar\delta}{C_1},\frac{1}{2C})$ and $\varepsilon<\frac{\delta}{2C}$, we find $\tilde\phi\in\Sigma$. This shows that $T$ is a map from $\Sigma$ to itself.

Now we prove that $T$ is a contraction map. Let $\phi_1,\phi_2\in\Sigma$. Let $\tilde\phi_i:=T\phi_i$ for $i=1,2$. By definition of $T$, we have:

$$\sum_{i,j=1,2}\big(a_{ij}^{\phi_k}\big(\tilde\phi_k-\phi_\infty\big)_{y_i}\big)_{y_j}=0,\quad k=1,2.$$

Taking the difference between the above two equations and letting $u:=\tilde\phi_1-\tilde\phi_2$, we find:

$$\sum_{i,j=1,2}\big(a_{ij}^{\phi_1}u_{y_i}\big)_{y_j}=-\sum_{i,j=1,2}\big(\big(a_{ij}^{\phi_1}-a_{ij}^{\phi_2}\big)\big(\tilde\phi_2-\phi_\infty\big)_{y_i}\big)_{y_j}, \tag{4.33}$$

with boundary condition $u|_{\partial R_f^Q}=0$. Again, by Lemma 4.2, we have:

$$\|u\|^{(-1-\alpha)}_{2,\alpha}\leqslant C\delta\big\|a_{ij}^{\phi_1}-a_{ij}^{\phi_2}\big\|^{(-\alpha)}_{1,\alpha}. \tag{4.34}$$

Define:

$$M_{ij}(y_2,s,\mathbf{r}) := N^i_{\phi_{y_j}}\big(A(y_2), B(y_2), (1-s)(0,\phi'_\infty)+s\mathbf{r}\big).$$

Hence, by definition of $a^\phi_{ij}$ in (4.31), we have:

$$a^{\phi_k}_{ij} = \int_0^1 M_{ij}(y_2,s,\nabla\phi_k)\,\mathrm{d}s. \tag{4.35}$$

Therefore, we obtain:

$$a^{\phi_1}_{ij} - a^{\phi_2}_{ij} = \int_0^1\int_0^1 \sum_{k=1,2} (M_{ij})_{r_k}\big(y_2,s,\nabla\phi_1+\mu\nabla(\phi_1-\phi_2)\big)(\phi_1-\phi_2)_{y_k}\,\mathrm{d}\mu\,\mathrm{d}s. \tag{4.36}$$

Then we have:

$$\big\|a^{\phi_1}_{ij} - a^{\phi_2}_{ij}\big\|^{(-\alpha)}_{1+\alpha} \leqslant C\|\phi_1-\phi_2\|^{(-1-\alpha)}_{2,\alpha}. \tag{4.37}$$

From (4.34)–(4.37), we obtain:

$$\big\|\tilde\phi_1-\tilde\phi_2\big\|^{(-1-\alpha)}_{2,\alpha} \leqslant C\delta\|\phi_1-\phi_2\|^{(-1-\alpha)}_{2,\alpha},$$

which implies that $T$ is a contraction map for sufficiently small $\delta<\delta_0$.

In terms of uniqueness, we consider two solutions $\phi_1$ and $\phi_2$ satisfying estimate (4.8). It is obvious that they are both fixed points of $T$. Since $T$ is a contraction map that admits a unique fixed point, we know that $\phi_1=\phi_2$. This completes the proof of Lemma 4.1. □

## 5. Fixed boundary problems in infinite nozzles

In this section, we study the solution in the unbounded domain:

$$R_f = \big\{\mathbf{y}:\ 0<y_2<M,\ y_1>f(y_2)\big\}. \tag{5.1}$$

We slightly change the boundary condition of Problem IV to form Problem V in the unbounded domain.

**Problem V.** For given incoming flow $\phi_-$ with

$$\left\|\phi_- - \frac{y_2}{M_0}\right\|_{2,\alpha;R_-} \leqslant \varepsilon,$$

and fixed shock front $\mathcal{F}=\{y_1=f(y_2)\}$, find a solution $\phi$ of the boundary value problem for Eq. (4.6) in $R_f$ with the boundary conditions:

$$\phi|_{\mathcal{F}} = \phi_-, \qquad \phi|_{\mathcal{B}_{0,1}} = \zeta_{0,1}.$$

Then we have:

**Lemma 5.1.** *Let the shock function $f=f(y_2)$ satisfy*:

$$\|f\|^{(-1-\alpha)}_{2,\alpha;(0,M)} \leqslant \delta$$

*for some small constant $\delta<\delta_0$, where $\delta_0$ is given in Lemma* 4.1. *Then there exists $\varepsilon_0>0$ such that, when $\varepsilon\in(0,\varepsilon_0)$, there exists a solution $\phi$ of Problem* V *with the estimate*:

$$\|\phi-\phi_\infty\|^{(-1-\alpha;\mathcal{B}_{0,1})}_{2,\alpha;R_f} \leqslant C\big(\varepsilon+\delta^2\big). \tag{5.2}$$

*Moreover, the following asymptotic behavior holds*:

$$\lim_{Q\to\infty}\|\phi-\phi_\infty\|^{(-1-\alpha)}_{2,\alpha;R_f\setminus R_f^Q} = 0. \tag{5.3}$$

*The solution is unique in the class of $\phi$ satisfying* (5.2).

**Proof.** We have estimate (4.8) for each solution $\phi_Q$ in $R_f^Q$ with $C$ independent of $Q$. To obtain the solution for Problem V, we simply let $Q \to \infty$. By choosing a proper subsequence $\{\phi_{Q_k}\} \subset \{\phi_Q\}$, $\phi_{Q_k}$ converges to a function $\phi$ in the $C^0$-norm on any compact subset of $\overline{R_f}$ and in the $C^2$-norm on any compact subset of $R_f$. The function $\phi$ is a solution of Problem V with estimate (5.2) directly from (4.8).

In terms of uniqueness, we need to investigate first the limiting behavior of $\phi$ as $y_1 \to \infty$. Define $\varphi := \phi - \phi_\infty$. Then $\varphi$ satisfies the following equation:

$$\sum_{i,j=1,2} \left(a_{ij}^{\phi} \varphi_{y_i}\right)_{y_j} = 0, \tag{5.4}$$

where $a_{ij}^{\phi}$ is defined in (4.31).

We multiply Eq. (5.4) by $\varphi$ and integrate over the finite domain $R_f^Q$. By integration by parts, we have:

$$\iint\limits_{R_f^Q} \sum_{i,j=1,2} a_{ij}^{\phi} \varphi_{y_i} \varphi_{y_j} \, \mathrm{d}\mathbf{y} = \int\limits_{\partial R_f^Q} \sum_{i,j=1,2} a_{ij}^{\phi} \varphi_{y_i} \varphi \nu_j \, \mathrm{d}s, \tag{5.5}$$

where $\nu_i$ is the $i$th component of the outer normal $\nu = (\nu_1, \nu_2)$, and d$s$ is the infinitesimal of arch length along the boundary of $R_f^Q$. By ellipticity of $(a_{ij}^{\phi})$, we know:

$$\iint\limits_{R_f^Q} |\nabla \varphi|^2 \, \mathrm{d}\mathbf{y} \leqslant C \iint\limits_{R_f^Q} \sum_{i,j=1,2} a_{ij}^{\phi} \varphi_{y_i} \varphi_{y_j} \, \mathrm{d}\mathbf{y}. \tag{5.6}$$

Assumption (2.11) implies that

$$\left|\varphi(y_1, \mathrm{i}M)\right| = |\zeta_i - \mathrm{i}| \leqslant \frac{\varepsilon}{(1+y_1)^{\beta}}, \quad i = 0, 1. \tag{5.7}$$

Together with the boundedness of $\nabla\varphi$, we conclude:

$$\int\limits_{\partial R_f^Q} \sum_{i,j=1,2} a_{ij}^{\phi} \varphi_{y_i} \varphi \nu_j \, \mathrm{d}s \leqslant C. \tag{5.8}$$

Hence, from (5.5)–(5.6) and (5.8), we obtain:

$$\iint\limits_{R_f^Q} |\nabla \varphi|^2 \, \mathrm{d}\mathbf{y} \leqslant C,$$

where $C$ is independent of $Q$. Therefore, letting $Q \to \infty$, we have:

$$\iint\limits_{R_f} |\nabla \varphi|^2 \, \mathrm{d}\mathbf{y} \leqslant C. \tag{5.9}$$

Let $D_Q := (Q-2, Q+2) \times (0, M)$. Then we have:

$$\lim_{Q\to\infty} \iint\limits_{D_Q} |\nabla \varphi|^2 \, \mathrm{d}\mathbf{y} = 0. \tag{5.10}$$

Since

$$\varphi(\mathbf{y}) = \zeta_0(y_1) + \int_0^{y_2} \varphi_{y_2}(y_1, s) \, \mathrm{d}s,$$

we obtain

$$\varphi^2 \leqslant 2{\zeta_0}^2 + 2M \int_0^M |\varphi_{y_2}|^2(y_1, s) \, \mathrm{d}s.$$

Hence,

$$\iint_{D_Q} \varphi^2 \,\mathrm{d}\mathbf{y} \leqslant 2M \int_{Q-2}^{Q+2} {\zeta_0}^2(y_1)\,\mathrm{d}y_1 + 2M^2 \iint_{D_Q} |\nabla\varphi|^2 \,\mathrm{d}\mathbf{y}.$$

By (5.10), we conclude:

$$\lim_{Q\to\infty} \iint_{D_Q} \varphi^2 \,\mathrm{d}\mathbf{y} = 0. \tag{5.11}$$

We then use the local estimates of $\varphi$ to control the $C^0$-norm of $\varphi$. From Theorem 8.25 in [16], we find that, for any $\mathbf{y} \in \{Q\} \times (0, M)$,

$$\|\varphi\|_{C^0(B_1(\mathbf{y}))} \leqslant C\big(\|\varphi\|_{C^0(B_2(\mathbf{y})\cap\mathcal{B}_{0,1})} + \|\varphi\|_{L^2(B_2(\mathbf{y}))}\big). \tag{5.12}$$

Estimates (5.11) and (5.12) imply that

$$\lim_{Q\to\infty} \|\phi - \phi_\infty\|_{C^0(R_f\cap\{y_1>Q\})} = 0.$$

Once we have the above decay in the $C^0$-norm, the asymptotic behavior (5.3) immediately follows by the Schauder interior estimates similar to Lemma 4.2 in the domain $(Q-1, Q+1) \times (0, M)$.

To obtain the uniqueness, we assume that there are two solutions $\phi_1$ and $\phi_2$ satisfying Eq. (3.27). By taking the difference of the two equations, we derive an elliptic equation for $\phi_1 - \phi_2$. By the asymptotic behavior of $\phi_1$ and $\phi_2$, we know that, for any small $\tau > 0$, there exists $Q$, depending on $\tau$, such that $|\phi_1(Q, y_2) - \phi_2(Q, y_2)| \leqslant \tau$. Other three boundary conditions for $\phi_1 - \phi_2$ are 0. Hence, the Maximum Principle implies that $\|\phi_1 - \phi_2\|_{C^0(R_f^Q)} \leqslant \tau$. Letting $\tau \to 0$ yields the uniqueness. □

## 6. Free boundary problems in infinite nozzles

Once we obtain the unique solution for the fixed shock in the infinite nozzle, we can update the location of the shock front by condition (3.8) and construct a map for the shock functions. The fixed point of this map is the real shock front. The process is the following:

Define the set for the shock iteration:

$$\mathcal{H} = \big\{f\colon\ \|f\|^{(-1-\alpha)}_{2,\alpha;(0,M)} \leqslant \delta\big\}. \tag{6.1}$$

For any $f \in \mathcal{H}$, we solve the fixed boundary problem, Problem V. We can express $U$ in terms of $\nabla\phi$ by (3.25). Then we use the Rankine–Hugoniot condition (3.8) to find a new shock $\tilde{f}$:

$$\tilde{f}'(y_2) = \frac{[u_2](f(y_2), y_2)}{[p](f(y_2), y_2)}, \tag{6.2}$$

with $\tilde{f}(0) = 0$.

We define the map $\mathcal{T}$ by $\tilde{f} = \mathcal{T} f$. We need to prove that $\mathcal{T}$ maps from $\mathcal{H}$ into itself and is also a contraction map.

First, by (6.2), we have:

$$\|\tilde{f}\|^{(-1-\alpha)}_{2,\alpha} \leqslant C\big(\|U_- - U_-^0\|_{1,\alpha;R_1} + \|U - U_+^0\|^{(-\alpha)}_{1,\alpha;R_f}\big).$$

Estimate (5.2) implies:

$$\|U - U_+^0\|^{(-\alpha)}_{1,\alpha;R_f} \leqslant C\left\|\phi - \frac{y_2}{M_0}\right\|^{(-1-\alpha)}_{2,\alpha;R_f} \leqslant C\big(\varepsilon + \delta^2\big).$$

Therefore, we obtain:

$$\|\tilde{f}\|^{(-1-\alpha)}_{2,\alpha} \leqslant C\big(\varepsilon + \delta^2\big) \leqslant \delta, \tag{6.3}$$

provided that $\delta < \delta_0$ and $\varepsilon < \frac{\delta}{2C}$. Hence, $\mathcal{T}$ is a map from $\mathcal{H}$ into itself.

Next, we prove that $\mathcal{T}$ is a contraction map. For any two shock functions $f_1, f_2 \in \mathcal{H}$, we solve Problem V to obtain the corresponding solutions $\phi_1, \phi_2$, respectively. Let $\tilde{f}_i = \mathcal{T} f_i$ for $i = 1, 2$.

In order to compare the two solutions $\phi_1$ and $\phi_2$, we map their own domains $R_{f_1}$ and $R_{f_2}$ to $R_0$ defined by:

$$R_0 = \{\bar{\mathbf{y}}\colon\ \bar{y}_1 > 0,\ 0 < \bar{y}_2 < M\}.$$

The coordinate transformations $\sigma_i : R_0 \to R_{f_i}$, $i = 1, 2$, are defined by:

$$\mathbf{y} = \sigma_i(\bar{\mathbf{y}}) = \big(\bar{y}_1 + \eta(\bar{y}_1) f_i(\bar{y}_2), \bar{y}_2\big), \tag{6.4}$$

where $\eta$ is a smooth function satisfying:

$$\eta(s) = \begin{cases} 0, & |s| > 1/2, \\ 1, & |s| < 1/4, \end{cases} \qquad |\eta|_{C^2} \leqslant 10. \tag{6.5}$$

A simple calculation gives:

$$\frac{\partial y_1}{\partial \bar{y}_1} = 1 + \eta' f_i, \qquad \frac{\partial y_1}{\partial \bar{y}_2} = \eta f_i', \qquad \frac{\partial y_2}{\partial \bar{y}_1} = 0, \qquad \frac{\partial y_2}{\partial \bar{y}_2} = 1.$$

Define $\bar{\phi}_i := \phi_i \circ \sigma_i$. When we change the coordinates from $\mathbf{y}$ to $\bar{\mathbf{y}}$ by $\sigma_k^{-1}$, Eq. (3.27) becomes:

$$\sum_{i=1,2} \big(\bar{N}^i\big(\bar{\mathbf{y}}, A_k, \nabla_{\bar{\mathbf{y}}}\bar{\phi}_k, f_k, f_k'\big)\big)_{\bar{y}_i} = 0, \tag{6.6}$$

where

$$\bar{N}^i\big(\bar{\mathbf{y}}, A_k, \nabla_{\bar{\mathbf{y}}}\bar{\phi}_k, f_k, f_k'\big) = \sum_{j=1,2} N^j\big(A_k, B, (\nabla_{\mathbf{y}}\phi_k) \circ \sigma_k\big) \frac{\partial \bar{y}_i}{\partial y_j} \left| \frac{\partial \mathbf{y}}{\partial \bar{\mathbf{y}}} \right|.$$

**Remark 6.1.** For a different fixed shock $y_1 = f(y_2)$, the coefficient $A(y_2)$ derived in (3.44) is different, depending on the shock location $y_1 = f(y_2)$. We denote $A_i$ the coefficient for the corresponding shock $f_i$. Also, the asymptotic behavior of $\phi_i$ depends on $f_i$ and is denoted by $\phi_\infty^{(i)}$.

For notational convenience, we set

$$\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2) = \nabla_{\mathbf{y}}\phi \circ \sigma, \qquad \mu = (\mu_1, \mu_2) = \nabla_{\bar{\mathbf{y}}}\bar{\phi},$$

where $\sigma$ is the coordinate transformation defined in (6.4) for a general shock $y_1 = f(y_2)$, $\phi$ is the solution in the $\mathbf{y}$-coordinates, and $\bar{\phi}$ is the solution in the $\bar{\mathbf{y}}$-coordinates. Hence, we have the relation:

$$\bar{\mathbf{t}} = \left(\frac{\mu_1}{1 + \eta' f}, \mu_2 - \frac{\mu_1 \eta f'}{1 + \eta' f}\right).$$

More explicit expressions for $\bar{N}^1$ and $\bar{N}^2$ are:

$$\bar{N}^1(\bar{\mathbf{y}}, A, \mu, f, f') = N^1\big(A, B, \bar{\mathbf{t}}\big) + N^2\big(A, B, \bar{\mathbf{t}}\big)(-\eta f'), \tag{6.7}$$

$$\bar{N}^2(\bar{\mathbf{y}}, A, \mu, f, f') = N^2\big(A, B, \bar{\mathbf{t}}\big)(1 + \eta' f). \tag{6.8}$$

Set $\xi = (A, \mu, f, f')$ and $\xi_i = (A_i, \nabla_{\bar{\mathbf{y}}}\bar{\phi}_i, f_i, f_i')$. Taking the difference of Eq. (6.6) for $k = 1, 2$ yields:

$$\sum_{i=1,2} \left( \int_0^1 \frac{\partial \bar{N}^i}{\partial \xi}\big(\bar{\mathbf{y}}, \xi_2 + s(\xi_1 - \xi_2)\big)\, \mathrm{d}s (\xi_1 - \xi_2) \right)_{\bar{y}_i} = 0. \tag{6.9}$$

Set:

$$\bar{a}_{ij} = \int_0^1 \frac{\partial \bar{N}^i}{\partial \mu_j}\big(\bar{\mathbf{y}}, \xi_2 + s(\xi_1 - \xi_2)\big)\, \mathrm{d}s, \qquad \bar{b}_i = \int_0^1 \frac{\partial \bar{N}^i}{\partial A}\big(\bar{\mathbf{y}}, \xi_2 + s(\xi_1 - \xi_2)\big)\, \mathrm{d}s,$$

$$\bar{c}_i = \int_0^1 \frac{\partial \bar{N}^i}{\partial f}\big(\bar{\mathbf{y}}, \xi_2 + s(\xi_1 - \xi_2)\big)\, \mathrm{d}s, \qquad \bar{d}_i = \int_0^1 \frac{\partial \bar{N}^i}{\partial f'}\big(\bar{\mathbf{y}}, \xi_2 + s(\xi_1 - \xi_2)\big)\, \mathrm{d}s.$$

Then we rewrite (6.9) into

$$\sum_{i,j=1,2} (\bar{a}_{ij}\bar{u}_{\bar{y}_i})_{\bar{y}_j} = \sum_{j=1,2} \big(\tilde{b}_i\big)_{\bar{y}_i}, \tag{6.10}$$

where

$$\bar{u} = \bar{\phi}_1 - \bar{\phi}_2, \qquad \tilde{b}_i = -\bar{b}_i(A_1 - A_2) - \bar{c}_i(f_1 - f_2) - \bar{d}_i(f_1' - f_2').$$

From the expressions of $\bar{N}^i$, it is easy to check that $(\bar{a}_{ij})$ satisfies the same ellipticity condition as in (4.16). Also, we have:

$$\|\bar{b}_i\|_{1,\alpha}^{(-\alpha)} \leqslant C, \qquad \big\|(\bar{c}_i, \bar{d}_i)\big\|_{1,\alpha}^{(-\alpha)} \leqslant C(\varepsilon + \delta).$$

We investigate the boundary conditions of $\bar{u}$. On the shock $\bar{y}_1 = 0$, we have:

$$\begin{aligned}\bar{u}(0, \bar{y}_2) &= \phi_-\big(f_1(\bar{y}_2), \bar{y}_2\big) - \phi_-\big(f_2(\bar{y}_2), \bar{y}_2\big)\\ &= \int_0^1 \frac{\partial \phi_-}{\partial y_1}\big(f_2 + s(f_1 - f_2)\big)\,\mathrm{d}s\,(f_1 - f_2).\end{aligned} \tag{6.11}$$

On the two walls $\mathcal{B}_{0,1}$, the condition is:

$$\begin{aligned}\bar{u}(\bar{y}_1, \mathrm{i}M) &= \zeta_i\big(\bar{y}_1 + \eta(\bar{y}_1) f_1(\mathrm{i}M)\big) - \zeta_i\big(\bar{y}_1 + \eta(\bar{y}_1) f_2(\mathrm{i}M)\big)\\ &= \int_0^1 \zeta_i'\big(\bar{y}_1 + s\eta(f_1 - f_2) + \eta f_2\big)\,\mathrm{d}s\ \eta\,(f_1 - f_2).\end{aligned} \tag{6.12}$$

We also truncate the domain $R_0$ with right end $\bar{y}_1 = Q$ and analyze the condition on $\bar{y}_1 = Q$. By the asymptotic behavior of $\phi_i$, we know that, for any given small constant $\tau > 0$, a large $Q$ can be chosen such that

$$\sum_{i=1,2} \|\phi_i(Q, \cdot) - \phi_\infty^{(i)}\|_{2,\alpha}^{(-1-\alpha)} \leqslant \tau.$$

From the derivation of $\phi_\infty$ in (3.47)–(3.48), we also have:

$$\|\phi_\infty^{(1)} - \phi_\infty^{(2)}\|_{2,\alpha}^{(-1-\alpha)} \leqslant C\|A_1 - A_2\|_{1,\alpha}^{(-\alpha)}.$$

On the other hand, from (3.45), we have:

$$\begin{aligned}\|A_1 - A_2\|_{1,\alpha}^{(-\alpha)} &\leqslant C\big(\|DU_-\|_{1,\alpha;R_-}\|f_1 - f_2\|_{1,\alpha} + \delta\|f_1' - f_2'\|_{1,\alpha}^{(-\alpha)}\big)\\ &\leqslant C(\varepsilon + \delta)\|f_1 - f_2\|_{2,\alpha}^{(-1-\alpha)}.\end{aligned}$$

Therefore, we obtain

$$\big\|\bar{u}(Q, \cdot)\big\|_{2,\alpha}^{(-1-\alpha)} \leqslant \tau + C(\varepsilon + \delta)\|f_1 - f_2\|_{2,\alpha}^{(-1-\alpha)}. \tag{6.13}$$

With the estimates on the boundary of $R_0^Q$, we apply Lemma 4.2 to obtain

$$\|\bar{u}\|_{2,\alpha;R_0^Q}^{(-1-\alpha)} \leqslant C\big(\tau + (\varepsilon + \delta)\|f_1 - f_2\|_{2,\alpha}^{(-1-\alpha)}\big). \tag{6.14}$$

Letting $\tau \to 0$ and $Q \to \infty$, we obtain the estimate:

$$\big\|\bar{\phi}_1 - \bar{\phi}_2\big\|_{2,\alpha;R_0}^{(-1-\alpha)} \leqslant C(\varepsilon + \delta)\|f_1 - f_2\|_{2,\alpha}^{(-1-\alpha)}. \tag{6.15}$$

The method that we update the new shocks $\tilde{f}_1, \tilde{f}_2$ in (6.2) indicates that

$$\|\tilde{f}_1 - \tilde{f}_2\|_{2,\alpha}^{(-1-\alpha)} \leqslant C\big((\varepsilon + \delta)\|f_1 - f_2\|_{2,\alpha}^{(-1-\alpha)} + \|\bar{\phi}_1 - \bar{\phi}_2\|_{2,\alpha;R_0}^{(-1-\alpha)}\big). \tag{6.16}$$

Estimates (6.15)–(6.16) immediately imply:

$$\|\tilde{f}_1 - \tilde{f}_2\|_{2,\alpha}^{(-1-\alpha)} \leqslant \frac{1}{2}\|f_1 - f_2\|_{2,\alpha}^{(-1-\alpha)} \tag{6.17}$$

for sufficiently small $\varepsilon, \delta > 0$. This is equivalent to saying that $\mathcal{T}$ is a contraction map. There exists a unique fixed point $f$ for $\mathcal{T}$, which is the real shock for Problem III. Then we fix this shock $f$ and solve a fixed boundary problem, Problem V, to obtain $\phi$. Through $\nabla\phi$, together with the quantities $A$ and $B$, we can find the solution $U$.

Estimates (3.18) and (3.19) are obtained by letting $\delta = 2C\varepsilon$, where $C$ is the same as that in estimate (5.2) in Lemma 5.1.

For uniqueness, we assume two solutions $(U_1, f_1)$ and $(U_2, f_2)$ satisfying condition (3.18). It is easy to see that $f_1$ and $f_2$ are two fixed points of $\mathcal{T}$. By contraction of $\mathcal{T}$, we conclude $f_1 = f_2$. Then, Lemma 5.1 gives the same potential $\phi_1 = \phi_2$. Since $U_i$ can be expressed by $\nabla\phi_i$ for $i = 1, 2$, $U_1$ and $U_2$ have to be equal. This concludes the uniqueness.

With these, the proof of Theorem 2.2 is completed.

## 7. Asymptotic behavior of solutions at infinity

After we obtain the solution of the Euler equations, the quantities $p_\infty$, $\phi_\infty(y_2)$, and $A(y_2)$ are determined by (3.47)–(3.49). They uniquely define the asymptotic limit $U_\infty$ of $U$ at the infinity, although $U_\infty$ is not given explicitly.

Assume further that the nozzle walls satisfy the decay condition:

$$\|\zeta_i - \mathrm{i}\|_{C^{1,\alpha}(x_1,\infty)} \leqslant \frac{\varepsilon}{(1+|x_1|)^\beta}, \tag{7.1}$$

for $x_1 > -1$, $i = 0, 1$. Then we can obtain the strong convergence of $U$ to $U_\infty$ with algebraic rate $x_1^{-\beta}$.

**Theorem 7.1.** *Under assumption* (7.1)*, the solution $U$ for Problem* I *converges to its asymptotic limit $U_\infty$ as $x_1 \to \infty$ with the following estimate*:

$$\|U - U_\infty\|_{1,\alpha;\Omega_X}^{(-\alpha;\Gamma_{0,1})} \leqslant \frac{C\varepsilon}{(1+|X|)^\beta}, \tag{7.2}$$

*where $\Omega_X = \Omega \cap \{x_1 > X\}$.*

**Proof.** To obtain estimate (7.2), it suffices to prove:

$$\|\phi - \phi_\infty\|_{2,\alpha;R_Y}^{(-1-\alpha;\mathcal{B}_{0,1})} \leqslant \frac{C\varepsilon}{(1+|Y|)^\beta}, \tag{7.3}$$

where $R_Y = (Y, \infty) \times (0, M)$, $\phi$ is the potential function defined in (3.20) and satisfies the nonlinear equation (4.30). The coefficients $a_{ij}^{\phi}$, $i, j = 1, 2$, satisfy condition (4.15). We first use a comparison function to obtain the $C^0$-estimate for $\phi - \phi_\infty$, and then the standard Schauder estimates lead to (7.3). The method is similar to that in Lemma 4.2.

To control $|\phi - \phi_\infty|$, we first estimate the solution of (4.30) in $R_f^Q$, denoted by $\phi_Q$, and then let $Q \to \infty$. We use the following comparison function:

$$v_3 = \frac{{y_2}^\alpha + (M - y_2)^\alpha}{(C_0 + y_1)^\beta}, \tag{7.4}$$

where $C_0$ is chosen large enough, depending only on the width of the nozzle $M$ and parameters $\alpha, \beta, \delta$. It is straightforward to check:

$$\sum_{i,j=1,2} \left(a_{ij}^{\phi_Q}(v_3)_{y_i}\right)_{y_j} \leqslant -\frac{\alpha(1-\alpha)}{2}(C_0 + y_1)^{-\beta}\left({y_2}^{\alpha-2} + (M - y_2)^{\alpha-2}\right) \leqslant 0, \tag{7.5}$$

for sufficiently large $C_0$, together with condition (4.15). Let $v_4 = C_1\varepsilon v_3$ with sufficiently large $C_1$. On the bounded domain $R_1^Q = (1, Q) \times (0, M)$, we use the boundary conditions (3.46) and (4.3). By (7.1), we know that $v_4 \geqslant \phi_Q - \phi_\infty$ on the walls $\mathcal{B}_{0,1}$. By the definition of $h_Q$, we have:

$$(\phi_Q - \phi_\infty)|_{y_1=Q} \leqslant \frac{C\varepsilon}{(1+Q)^\beta} \leqslant v_4|_{y_1=Q}.$$

Also estimate (4.8) implies $(\phi_Q - \phi_\infty)|_{y_1=1} \leqslant C\varepsilon \leqslant v_4|_{y_1=1}$. By the Comparison Principle, we conclude:

$$\left|\phi_Q(\mathbf{y}) - \phi_\infty(y_2)\right| \leqslant \frac{C\varepsilon}{(1+|y_1|)^\beta}. \tag{7.6}$$

Since the constant $C$ in (7.6) is independent of $Q$, we let $Q$ tend to infinity to obtain the same estimate for $\phi$.

Once we have the $C^0$-estimate of $|\phi_Q - \phi_\infty|$ above, we use the standard Schauder boundary estimate (up to the boundary $\mathcal{B}_{0,1}$) in the domain,

$$R_{Y-2}^{Y+2} := (Y-2, Y+2) \times (0, M),$$

to obtain:

$$\|\phi - \phi_\infty\|_{2,\alpha;R_{Y-1}^{Y+1}}^{(-1-\alpha;\mathcal{B}_{0,1})} \leqslant \frac{C\varepsilon}{(1+|Y|)^\beta}.$$

This immediately gives estimate (7.3). $\square$

## 8. Shock stability

In this section, we discuss the stability of transonic shocks under the small perturbation of both the incoming flows and the nozzle walls. We first investigate the shock stability in Lagrangian coordinates.

### *8.1. Shock stability in Lagrangian coordinates*

Assume that there are two nozzles whose walls are described by $\zeta_i^{(1)}$ and $\zeta_i^{(2)}$, $i = 0, 1$, and two incoming flows $U_-^{(i)}$, $i = 1, 2$, in Eulerian coordinates.

Let:

$$M_i = \int_{\zeta_0^{(i)}(-1)}^{\zeta_1^{(i)}(-1)} \rho_-^{(i)} u_{1-}^{(i)}(-1, s)\, \mathrm{d}s, \quad i = 1, 2. \tag{8.1}$$

The domains for the transonic flows $U_-^{(i)}$ and $U^{(i)}$ in Lagrangian coordinates are defined by:

$$R^{(i)} = (-1, \infty) \times (0, M_i). \tag{8.2}$$

We use $R^{(0)}$ as our standard domain to compare the flows $U^{(1)}$ and $U^{(2)}$ later, where $M_0$ is defined in (3.10). The domains for the incoming flows $U_-^{(i)}$ are:

$$R_1^{(i)} := (-1, 1) \times (0, M_i), \tag{8.3}$$

and the domains for the subsonic flows $U^{(i)}$ are:

$$R_{f_i}^{(i)} := R^{(i)} \cap \left\{y_1 > f_i(y_2)\right\}, \tag{8.4}$$

where $f_i$ are the corresponding shock functions.

To compare the two different flows, we need to use the standard domain $R^{(0)}$. Hence, we define the coordinate transformations $\bar{\sigma}_i : R^{(0)} \to R^{(i)}$ by:

$$\mathbf{y} = \bar{\sigma}_i(\bar{\mathbf{y}}) = \left(\bar{y}_1 + \eta(\bar{y}_1)\bar{f}_i(\bar{y}_2), \frac{M_i}{M_0}\bar{y}_2\right), \tag{8.5}$$

where

$$\bar{f}_i(\bar{y}_2) = f_i\left(\frac{M_i}{M_0}\bar{y}_2\right), \tag{8.6}$$

and $\eta$ is defined in (6.5). Therefore, $\bar{\sigma}_i^{-1}$, $i = 1, 2$, map the domains $R^{(i)}$ into $R^{(0)}$ and flatten out the shocks $f_i$. Also, we define:

$$\bar{U}_-^{(i)}(\bar{\mathbf{y}}) := U_-^{(i)}\left(\bar{y}_1, \frac{M_i}{M_0}\bar{y}_2\right). \tag{8.7}$$

We now state the theorem for the shock stability in Lagrangian coordinates.

**Theorem 8.1.** *Let two incoming flows $U_-^{(i)}$, $i = 1, 2$, in two nozzles $R^{(i)}$ satisfy*:

$$\|U_-^{(i)} - U_-^0\|_{C^{2,\alpha}(R_1^{(i)})} \leqslant \varepsilon. \tag{8.8}$$

*Let $U^{(i)}$ and $f_i$ be the unique solution and corresponding shock for each $i = 1, 2$, determined by Theorem* 3.1. *Then, in the domain $R^{(0)}$, we have the following stability estimate*:

$$\|\bar{f}_1 - \bar{f}_2\|_{2,\alpha;(0,M_0)}^{(-1-\alpha)} \leqslant C\left(\|\bar{U}_-^{(1)} - \bar{U}_-^{(2)}\|_{1,\alpha;R_1^{(0)}}^{(-\alpha)} + \sum_{i=0,1}\|\zeta_i^{(1)} - \zeta_i^{(2)}\|_{1,\alpha;(-1,\infty)}\right). \tag{8.9}$$

**Proof.** The proof basically follows the procedure in Section 6. The difference between the transformations $\bar{\sigma}_i$ here and $\sigma_i$ in Section 6 is that $\bar{\sigma}_i$ also scale the vertical direction by the factor $M_i/M_0$, $i = 1, 2$. Therefore, there is an extra term $M_i$ appearing in $\bar{N}^i$ as in (6.6).

Also notice the difference in the upper and lower boundary conditions. We obtain:

$$\|\bar{f}_1 - \bar{f}_2\|_{2,\alpha;(0,M_0)}^{(-1-\alpha)} \leqslant C\left(\|\bar{U}_-^{(1)} - \bar{U}_-^{(2)}\|_{1,\alpha;R_1^{(0)}}^{(-\alpha)} + |M_1 - M_2| + \sum_{i=0,1}\|\zeta_i^{(1)} - \zeta_i^{(2)}\|_{1,\alpha;(-1,\infty)}\right).$$

It is easy to check:

$$|M_1 - M_2| \leqslant C\left(\|\bar{U}_-^{(1)} - \bar{U}_-^{(2)}\|_{0,0;R_1^{(0)}} + \sum_{i=0,1}\|\zeta_i^{(1)} - \zeta_i^{(2)}\|_{0,0;(-1,\infty)}\right).$$

Thus, estimate (8.9) is obtained from the above inequalities. $\square$

*8.2. Shock stability in Eulerian coordinates*

The stability in Eulerian coordinates is not as nice as in Lagrangian coordinates due to the fact that the solutions and the shocks are not smooth enough at the corners. Before we describe the shock stability in Eulerian coordinates, we introduce some notations and transformations, similarly as in the Lagrangian case.

Let two nozzles $\Omega^{(1)}$ and $\Omega^{(2)}$ be defined by:

$$\Omega^{(i)} := \left\{\mathbf{x}\colon -1 < x_1 < \infty,\ \zeta_0^{(i)}(x_1) < x_2 < \zeta_1^{(i)}(x_1)\right\}. \tag{8.10}$$

The domains for the incoming flows $U_-^{(i)}$ are:

$$\Omega_1^{(i)} := \Omega^{(i)} \cap \{x_1 < 1\}. \tag{8.11}$$

We also define the standard flat nozzle and its entrance section:

$$\Omega^{(0)} := (-1, \infty) \times (0, 1), \qquad \Omega_1^{(0)} := (-1, 1) \times (0, 1). \tag{8.12}$$

Define $\pi_i : \Omega^{(i)} \to \Omega^{(0)}$ by:

$$\bar{\mathbf{x}} := \pi_i(\mathbf{x}) = \left(x_1, \frac{x_2 - \zeta_0^{(i)}(x_1)}{\zeta_1^{(i)}(x_1) - \zeta_0^{(i)}(x_1)}\right). \tag{8.13}$$

Let $\bar{U}_-^{(i)} := U_-^{(i)} \circ \pi_i^{-1}$. Let the $x_2$-coordinates of the lower and upper ends of the shock $S_i = \{x_1 = s_i(x_2)\}$ be $z_i$ and $w_i$ respectively. We rescale the shocks $s_i$ to $\bar{s}_i$ onto $(0, 1)$ by:

$$\bar{s}_i(\bar{x}_2) = s_i\big(z_i + (w_i - z_i)\bar{x}_2\big). \tag{8.14}$$

We have the following stability result in the $C^\alpha$-norm.

**Theorem 8.2.** *Let the two incoming flows $U_-^{(1)}$ and $U_-^{(2)}$ be defined in $\Omega_1^{(1)}$ and $\Omega_1^{(2)}$ respectively, and satisfy:*

$$\left\| U_-^{(i)} - U_-^0 \right\|_{C^{2,\alpha}(\Omega_1^{(i)})} \leqslant \varepsilon, \quad i = 1, 2. \tag{8.15}$$

*Let $U^{(i)}$ and $s_i$ be the unique solution and corresponding shock for each $i = 1, 2$, determined by Theorem 2.2. Then*

$$\|\bar{s}_1 - \bar{s}_2\|_{0,\alpha;(0,1)} \leqslant C\left( \left\| \bar{U}_-^{(1)} - \bar{U}_-^{(2)} \right\|^{(-\alpha)}_{1,\alpha;\Omega_1^{(0)}} + \sum_{i=0,1} \left\| \zeta_i^{(1)} - \zeta_i^{(2)} \right\|_{1,\alpha;(-1,\infty)} \right). \tag{8.16}$$

**Proof.** We notice that the shock $y_1 = f_i(y_2)$ in Lagrangian coordinates gives an equation for $\bar{s}_i$ as below:

$$\bar{s}_i(\bar{x}_2) = \bar{f}_i\left( \frac{M_0}{M_i} F_i\big(\bar{s}_i(\bar{x}_2), z_i + (w_i - z_i)\bar{x}_2\big) \right), \tag{8.17}$$

where

$$F_i(\mathbf{x}) = \int_{\zeta_0^{(i)}(x_1)}^{x_2} \rho_-^{(i)} u_{1-}^{(i)}(x_1, s)\, \mathrm{d}s.$$

Using (8.17), we find:

$$\begin{aligned} \|\bar{s}_1 - \bar{s}_2\|_{0,\alpha;(0,1)} \leqslant{}& C \left\| \bar{f}_1 - \bar{f}_2 \right\|_{0,\alpha;(0,M_0)} \\ &+ C\varepsilon\big( |M_1 - M_2| + |z_1 - z_2| + |w_1 - w_2| + \|\bar{s}_1 - \bar{s}_2\|_{0,\alpha;(0,1)} + \left\| \bar{U}_-^{(1)} - \bar{U}_-^{(2)} \right\|^{(-\alpha)}_{1,\alpha;\Omega_1^{(0)}} \big). \end{aligned}$$

Since $z_i = \zeta_0^{(i)}(\bar{s}_i(0))$ and $w_i = \zeta_1^{(i)}(\bar{s}_i(1))$, we conclude:

$$|z_1 - z_2| + |w_1 - w_2| \leqslant C\left( \sum_{i=0,1} \left\| \zeta_i^{(1)} - \zeta_i^{(2)} \right\|_{0,0;(-1,\infty)} + \|\bar{s}_1 - \bar{s}_2\|_{0,0;(0,1)} \right).$$

Therefore, applying (8.9), we obtain the $C^\alpha$-stability (8.16) for the shocks in Eulerian coordinates. □

## 9. Solutions in the supersonic region

In this section we give the proof of Theorem 2.1, regarding the existence and uniqueness of supersonic solutions in the upstream region $\Omega_1$ defined in (2.3). We actually solve an initial-boundary value problem (2.12)–(2.13) and (2.14)–(2.15) for the quasilinear hyperbolic system (1.1).

**Proof of Theorem 2.1.** We employ the characteristic method and the contraction mapping theorem to solve the initial-boundary value problem for the hyperbolic system.

Although the proof can be done for the Euler system in the original coordinates, we prefer to use the Lagrangian coordinates, in which case the boundaries are straight and hence the proof is more straightforward. Since the coordinate transformation is $C^{3,\alpha}$, we can go back to the original coordinates with the same smoothness and $C^{2,\alpha}$-estimates.

We rewrite the Euler equations (3.2)–(3.4) in Lagrangian coordinates as the non-divergence form:

$$AV_{y_1} + BV_{y_2} = 0, \tag{9.1}$$

where $V = (\mathbf{u}, p)^\top$,

$$A = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ \frac{1}{\rho u_1} & 0 & \frac{1}{\rho^2 c^2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & -\frac{u_2}{u_1} \\ 0 & 0 & 1 \\ -\frac{u_2}{u_1} & 1 & 0 \end{pmatrix}, \tag{9.2}$$

and $\rho$ can be expressed by $V$ through Bernoulli's law (3.21).

By solving $\det(B - \lambda A) = 0$, we find the eigenvalues of (9.1):

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \frac{c\rho(cu_2 \pm u_1\sqrt{u_1{}^2 + u_2{}^2 - c^2})}{u_1{}^2 - c^2}.$$

The corresponding left eigenvectors are:

$$l_1 = (u_1, u_2, 0), \qquad l_i = \left(-\frac{u_2}{u_1} - \frac{\lambda_i}{\rho u_1}, 1, \lambda_i\right), \quad i = 2, 3.$$

System (9.1) is strictly hyperbolic when $u_1 > c$. We employ the characteristic method to solve the problem. The approach is standard (cf. [19]).

The initial-boundary value problem for the hyperbolic system is more subtle than the Cauchy problem because the well-posedness of the initial-boundary value problem requires restrictive boundary conditions. In our case, we do have the proper boundary condition, which is the slip condition (2.13). For a given point in the domain, the characteristic starting from this point may hit the boundary when it travels backward. Since one slip condition is not enough to determine the values of $V$ of three variables on the boundary, we need to trace the information back from the initial data by two of the characteristics: one travels along the wall and the other travels back inside the domain. The immediate conclusion from some estimates and the contraction mapping is the local existence and uniqueness of the solution. If the initial data is a small perturbation from the constant state, the lifespan of the solution can be long enough. We refer the reader to Section 6 in [4] for more detailed discussion. This completes the proof. □

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